<u>Note:</u> **The 2nd exam will take place on Thursday, February 4, 8.15 am, in room A010 during the tutorial. The result will be available on the net shortly after:** *http://homepages.physik.uni-muenchen.de/~Michael.Haack//ST1_WS09.html* **. There will still be classes on February 8 and 10 and a tutorial on February 11.**

### Exercise 1 – Nilpotency of the BRST charge

In class we discussed the general form of the BRST charge

$$Q = c^i \left( K_i - \frac{1}{2} f_{ij}{}^k c^j b_k \right) \ . \tag{1}$$

Use the algebra of the generators of the gauge symmetry

$$[K_i, K_j] = f_{ij}{}^k K_k \tag{2}$$

and the anticommutator of the ghosts

$$\{b_j, c^i\} = \delta^i{}_j \tag{3}$$

to show $Q^2 = \frac{1}{2}\{Q, Q\} = 0$. To do so, you will need to use the Jacobi identity $f_{[ij}{}^k f_{m]k}{}^n = 0$, where the brackets indicate antisymmetrization of the enclosed indices.

### Exercise 2 – Green's function on the disk

The Green's function $G(z, \bar{z}, z', \bar{z}') = \langle X(z, \bar{z}) X(z', \bar{z}') \rangle$ on the disk (i.e. the upper half plane compactified by adding a point at infinity) obeys (for $z'$ away from the boundary)

$$\partial_z \partial_{\bar{z}} G = -\pi \alpha' \delta^2(z - z') \tag{4}$$

and

$$\partial_\sigma G|_{\sigma=0} = 0 \tag{5}$$

for Neumann boundary conditions.[1] Boundary problems like (4) & (5) are solved in electrodynamics with the help of the method of image charges. Using

$$\partial_z \partial_{\bar{z}} \ln |z|^2 = 2\pi \delta^2(z), \tag{6}$$

show by introducing an appropriate image charge that the Green's function is given by

$$G(z, \bar{z}, z', \bar{z}') = -\frac{\alpha'}{2} \ln |z - z'|^2 - \frac{\alpha'}{2} \ln |z - \bar{z}'|^2 \ . \tag{7}$$

Verify explicitly that the boundary condition (5) is fulfilled. To do so, express it in terms of derivatives with respect to $z$ and $\bar{z}$.

[1] We neglected the background charge in (4), i.e. strictly speaking we are calculating the Green's function on the non-compact upper half plane instead of the compact disk; however, as for the sphere, the difference is irrelevant for calculating disk amplitudes.

**Exercise 3 – The 1-loop vacuum amplitude**

The (1-loop) contribution to the vacuum energy in oriented closed string theory is given by the torus amplitude without any vertex operators. In order to discuss this amplitude, let us start by considering the vacuum energy in QFT, in particular for the field theory of a scalar with mass $M$ in $D$ dimensions described by the (Euclidean) action

$$S = \int d^D x \left( \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi + \tfrac{1}{2} M^2 \phi^2 \right) \ . \tag{8}$$

The vacuum energy $\Gamma$ is defined via the path integral

$$e^{-\Gamma} = \int \mathcal{D}\phi \ e^{-S} \sim \det{}^{-1/2} \left( -\Delta + M^2 \right) \ . \tag{9}$$

This can be further rewritten by using the identity

$$\ln(\det(A)) = -\int_\epsilon^\infty \frac{dt}{t} \mathrm{tr} \left( e^{-tA} \right) \ , \tag{10}$$

where $\epsilon$ is an ultraviolet cutoff and $t$ is a Schwinger parameter. The kinetic operator can be diagonalized by the complete set of momentum eigenstates, which results in

$$\Gamma = -\frac{V}{2} \int_\epsilon^\infty \frac{dt}{t} e^{-tM^2} \int \frac{d^D p}{(2\pi)^D} e^{-tp^2} \ , \tag{11}$$

where the space-time volume $V$ arises from the continuum normalization of the momentum, i.e. $\Sigma_p$ becomes $V(2\pi)^{-D} \int d^D p$. Performing the Gaussian momentum integrals yields

$$\Gamma = -\frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}} e^{-tM^2} \ . \tag{12}$$

For several bosonic fields, one has to sum over their contributions, i.e.

$$\Gamma_{\mathrm{tot}} = -\frac{V}{2(4\pi)^{D/2}} \int_\epsilon^\infty \frac{dt}{t^{D/2+1}} \mathrm{tr} \left( e^{-tM^2} \right) \ , \tag{13}$$

where the trace is over the entire mass spectrum.

One can now try to apply (13) to the bosonic string in $D = 26$, whose mass spectrum is encoded in

$$M^2 = \frac{2}{\alpha'} (L_0 + \tilde{L}_0 - 2) \ , \tag{14}$$

subject to the level matching condition $L_0 = \tilde{L}_0$.

**a)** Impose the level matching condition by using the integral representation of the Kronecker delta

$$\int_{-1/2}^{1/2} ds \ e^{2\pi i s (L_0 - \tilde{L}_0)} = \delta_{L_0, \tilde{L}_0} \ . \tag{15}$$

Show that the result can be written as

$$\Gamma_{\mathrm{tot}} = -\frac{V}{2(4\pi^2 \alpha')^{13}} \int_{-1/2}^{1/2} d\tau_1 \int_\epsilon^\infty \frac{d\tau_2}{\tau_2^{14}} \ \mathrm{tr} \left( q^{L_0 - 1} \bar{q}^{\tilde{L}_0 - 1} \right) \ , \tag{16}$$

where we introduced the complex Schwinger parameter

$$\tau = \tau_1 + i\tau_2 = s + i\frac{t}{\alpha'\pi} \tag{17}$$

and the notation

$$q = e^{2\pi i\tau} \quad , \quad \bar{q} = e^{-2\pi i\bar{\tau}} \; . \tag{18}$$

This formula shows an ultraviolet divergence for $\epsilon \to 0$, as usual for field theory. In string theory, an exact treatment would lead to a similar formula as (16) with the $\tau$ of (17) given by the modulus of the world-sheet torus. As discussed in class, however, the torus modulus should only be integrated over the fundamental domain $\mathcal{F}_0$, which introduces an effective ultraviolet cutoff!

**b)** Up to an overall factor, the torus amplitude without vertex operators is, thus, given by

$$\mathcal{T} = \int_{\mathcal{F}_0} \frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2^{12}}\mathrm{tr}\left(q^{L_0-1}\bar{q}^{\tilde{L}_0-1}\right) \; . \tag{19}$$

Perform the trace over the bosonic string spectrum by using

$$L_0 = \sum_{n=1}^{\infty} n\, a_n^\dagger a_n \tag{20}$$

with $a_n^\dagger a_n$ the number operator of a harmonic oscillator with frequency $n$, and similarly for $\tilde{L}_0$. You should obtain

$$\mathcal{T} = \int_{\mathcal{F}_0} \frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2^{12}}\frac{1}{|\eta(\tau)|^{48}} \; , \tag{21}$$

where

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \tag{22}$$

is the *Dedekind* $\eta$ function.

**c)** Show that both, the measure $\tau_2^{-2}d^2\tau$ as well as the integrand $\tau_2^{-12}|\eta(\tau)|^{-48}$ of (21), are invariant under an $SL(2,\mathbb{Z})$ transformation

$$\tau \to \tau' = \frac{a\tau+b}{c\tau+d} \quad , \quad a,b,c,d \in \mathbb{Z} \quad , \quad ad-bc = 1 \; . \tag{23}$$

To do so, show that

$$d^2\tau \to \frac{d^2\tau}{|c\tau+d|^4} \quad , \quad \tau_2 \to \frac{\tau_2}{|c\tau+d|^2} \tag{24}$$

and use the transformation of the Dedekind $\eta$ function under the generators $T$ and $S$

$$T: \eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau) \quad , \quad S: \eta(-1/\tau) = \sqrt{-i\tau}\;\eta(\tau) \; . \tag{25}$$