Exercise 1 – More constraints

a) *Point particle revisited:* In class we discussed the point particle action

$$S = \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau e\Big(e^{-2}\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} - m^2\Big) \ . \tag{1}$$

What are the primary constraints and the total Hamiltonian? Check consistency of the primary constraints with the equations of motion and see whether you get any secondary constraints. Which of the constraints are first class and which are second class. What is the extended Hamiltonian in this case?

b) *First class constraints:* Consider the matrix of the Poisson-brackets of all the constraints,

$$C_{ij} = [\phi_i, \phi_j]_{PB} \ . \tag{2}$$

Show: If $\det(C_{ij}) \approx 0$, there exists (at least) one first-class constraint among the constraints.

Exercise 2 – Longitudinal waves on non-relativistic strings

Consider a string with uniform mass density μ stretched between $x = 0$ and $x = a$ and with equilibrium tension T_0. Longitudinal waves are possible if the tension of the string varies as it stretches or compresses. For a piece of this string with equilibrium length L, a small change ΔL of its length is accompanied by a small change of the tension

$$\Delta T = \tau_0 \frac{\Delta L}{L} \ , \tag{3}$$

where τ_0 is a tension coefficient. Find the equation governing the small longitudinal oscillations of this string. Give the velocity of the waves.

Exercise 3 – Induced metric on S^2 from stereographic parameterization

Consider a unit sphere S^2 in $\mathbb{R}^3$ centered at the origin: $x^2 + y^2 + z^2 = 1$. In the stereographic parameterization of the sphere one uses parameters ξ^1 and ξ^2 to parameterize a point on the sphere, i.e.

$$\vec{x}(\xi^1, \xi^2) = \Big(x(\xi^1, \xi^2), y(\xi^1, \xi^2), z(\xi^1, \xi^2)\Big) \ . \tag{4}$$

Given parameters (ξ^1, ξ^2), the corresponding point on the sphere is that which lies on the line that goes through the north pole $N = (0, 0, 1)$ and the point $(\xi^1, \xi^2, 0)$.

a) Draw a sketch for the above construction. What are the required ranges for ξ^1 and ξ^2 if we wish to parameterize the full sphere?

b) Calculate the functions $x(\xi^1, \xi^2), y(\xi^1, \xi^2)$ and $z(\xi^1, \xi^2)$.

c) Calculate the four components of the induced metric

$$g_{ij}(\xi) = \frac{\partial \vec{x}}{\partial \xi^i} \cdot \frac{\partial \vec{x}}{\partial \xi^j} \ . \tag{5}$$

The algebra is a bit messy, but the result is simple. You can use a computer program like Maple or Mathematica for this part.

d) Check your result by computing the area of the sphere using the formula from class:

$$A = \int d\xi^1 d\xi^2 \sqrt{g} \ . \tag{6}$$

Exercise 4 – Reparameterization invariance of the area

Show that the area defined by (6) is invariant under a change of variables

$$\xi^i \longrightarrow \tilde{\xi}^i = \tilde{\xi}^i(\xi^1, \xi^2) \ . \tag{7}$$