Exercise 1 – p-brane action

The Nambu-Goto action of the string can be generalized to describe the dynamics of a p-brane, sweeping out a $(p+1)$-dimensional world volume, i.e.

$$S_p^{(\mathrm{NG})} = -T_p \int d^{p+1}\sigma \sqrt{-\det(\gamma_{\alpha\beta})} \ , \tag{1}$$

where $\gamma_{\alpha\beta} = \eta_{\mu\nu}\,\partial_\alpha X^\mu \partial_\beta X^\nu$, $\alpha, \beta = 0, \ldots, p$ is the induced metric on the p-brane. Show that the Polyakov form of the action requires a non-vanishing cosmological constant Λ_p for $p \neq 1$, i.e.

$$S_p^{(\mathrm{Pol})} = -\frac{T_p}{2} \int d^{p+1}\sigma \sqrt{-g}\, g^{\alpha\beta}\, \eta_{\mu\nu}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu + \Lambda_p \int d^{p+1}\sigma \sqrt{-g} \ . \tag{2}$$

Hint: First argue that the equations of motion for the X^μ derived from $S_p^{(\mathrm{NG})}$ and $S_p^{(\mathrm{Pol})}$ only agree if $g_{\alpha\beta}$ is given (up to a constant overall factor) by the induced metric

$$g_{\alpha\beta} = \eta_{\mu\nu}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu \ . \tag{3}$$

Then demand that the equations of motion for $g_{\alpha\beta}$, following from $S_p^{(\mathrm{Pol})}$, have (3) as a solution.

Exercise 2 – Closed string solution

Consider a closed string with initial conditions

$$\begin{aligned}
X^0(\tau=0,\sigma) &= 0 \quad , \quad \dot{X}^0(\tau=0,\sigma) = R \ , \\
X^1(\tau=0,\sigma) &= R\cos(\sigma) \quad , \quad \dot{X}^1(\tau=0,\sigma) = 0 \ , \\
X^2(\tau=0,\sigma) &= R\sin(\sigma) \quad , \quad \dot{X}^2(\tau=0,\sigma) = 0 \ , \\
X^i(\tau=0,\sigma) &= 0 \quad , \quad \dot{X}^i(\tau=0,\sigma) = 0 \quad , \quad i = 3, \ldots, D-1 \ ,
\end{aligned} \tag{4}$$

i.e. at $\tau = 0$ it is a circular string of radius R at rest, localized in the (X^1, X^2)-plane. Determine its time evolution by solving the equations of motion and show that the solution solves the constraints $\dot{X} \cdot X'$ and $\dot{X}^2 + X'^2 = 0$.

Exercise 3 – Virasoro algebra

a) The canonical momentum densities of the Polyakov action in conformal gauge are given by

$$P^\mu(\tau,\sigma) = T\dot{X}^\mu \ . \tag{5}$$

Use the expansion of the coordinates X^μ in Fourier modes, i.e.

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left[\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right] \tag{6}$$

and the Poisson brackets

$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')]_{\mathrm{PB}} = [P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')]_{\mathrm{PB}} = 0 \ , \tag{7}$$

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')]_{\mathrm{PB}} = \eta^{\mu\nu}\delta(\sigma-\sigma') \tag{8}$$

to show that

$$[\alpha^\mu_m, \alpha^\nu_n]_{\rm PB} = [\tilde\alpha^\mu_m, \tilde\alpha^\nu_n]_{\rm PB} = -im\eta^{\mu\nu}\delta_{m+n,0} \quad , \quad [\tilde\alpha^\mu_m, \alpha^\nu_n]_{\rm PB} = 0 \quad , \quad m,n \in \mathbb{Z} \ . \tag{9}$$

Hint: First show that

$$\begin{aligned} \Big[(\dot X^\mu \pm X^{\mu\prime})(\tau,\sigma), (\dot X^\nu \pm X^{\nu\prime})(\tau,\sigma')\Big]_{\rm PB} &= \pm 2T^{-1}\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') \ , \\ \Big[(\dot X^\mu \pm X^{\mu\prime})(\tau,\sigma), (\dot X^\nu \mp X^{\nu\prime})(\tau,\sigma')\Big]_{\rm PB} &= 0 \ . \end{aligned} \tag{10}$$

Use this to verify (9) (remember $\alpha^\mu_0 = \tilde\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu$).

b) The Virasoro generators of the closed string are given by

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \quad , \quad \tilde L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\tilde\alpha_{m-n}\cdot\tilde\alpha_n \ . \tag{11}$$

Use the result of part a) to show that they satisfy the (*Virasoro*) algebra

$$\begin{aligned} [L_m, L_n]_{\rm PB} &= -i(m-n)L_{m+n} \ , \\ [\tilde L_m, \tilde L_n]_{\rm PB} &= -i(m-n)\tilde L_{m+n} \ , \\ [L_m, \tilde L_n]_{\rm PB} &= 0 \ . \end{aligned} \tag{12}$$