Exercise 1 – Virasoro modes as generators of conformal transformations

In class we discussed the fact that the Virasoro modes generate conformal transformations via the Poisson bracket. Show

$$[X^\mu(\sigma^+,\sigma^-), L_n]_{\mathrm{PB}} = e^{in\sigma^-}\frac{\partial}{\partial\sigma^-}X^\mu(\sigma^+,\sigma^-)\ . \tag{1}$$

To do so, in addition to the Poisson brackets

$$[\alpha^\mu_m,\alpha^\nu_n]_{\mathrm{PB}} = [\tilde\alpha^\mu_m,\tilde\alpha^\nu_n]_{\mathrm{PB}} = -im\eta^{\mu\nu}\delta_{m+n,0}\quad,\quad [\tilde\alpha^\mu_m,\alpha^\nu_n]_{\mathrm{PB}} = 0\quad,\quad m,n\in\mathbb{Z}\ , \tag{2}$$

known from the last exercise sheet, you will also need the relations

$$[\alpha^\mu_n, x^\nu]_{\mathrm{PB}} = 0\quad,\quad n\neq 0\quad,\quad [x^\mu,p^\nu]_{\mathrm{PB}} = \eta^{\mu\nu}\ . \tag{3}$$

These can be derived in a similar way as in exercise 3 a) of assignment sheet 3, but they do not have to be proven here.

Exercise 2 – Angular momentum generators

Show that for a closed string the angular momentum generators

$$J^{\mu\nu} = T\int_0^{2\pi} d\sigma\,(X^\mu\,\partial_\tau X^\nu - X^\nu\,\partial_\tau X^\mu) \tag{4}$$

are given by

$$J^{\mu\nu} = x^\mu\,p^\nu - x^\nu\,p^\mu - i\sum_{n\neq 0}\frac{1}{n}\,(\alpha^\mu_{-n}\,\alpha^\nu_n + \tilde\alpha^\mu_{-n}\,\tilde\alpha^\nu_n)\ . \tag{5}$$

Exercise 3 – DD and ND boundary conditions for open strings

a) Consider an open string coordinate X^a with Dirichlet boundary conditions on both sides, i.e.

$$X^a(\tau,0) = x^a_1\quad,\quad X^a(\tau,\pi) = x^a_2\ . \tag{6}$$

Show that the most general solution to the open string equations of motion with these boundary conditions can be expanded as

$$X^a = x^a_1 + \frac{x^a_2 - x^a_1}{\pi}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^a_n e^{-in\tau}\sin(n\sigma) \tag{7}$$

with $\alpha^a_{-n} = (\alpha^a_n)^*$.

b) Now consider an open string coordinate X^r with Neumann boundary conditions on one side and Dirichlet boundary conditions on the other side, i.e.

$$\frac{\partial X^r}{\partial\sigma}(\tau,0) = 0\quad,\quad X^r(\tau,\pi) = x^r_2\ . \tag{8}$$

Show that the most general solution to the open string equations of motion with these boundary conditions can be expanded as

$$X^r = x^r_2 + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}_{\mathrm{odd}}}\frac{2}{n}\alpha^r_{\frac{n}{2}}e^{-i\frac{n}{2}\tau}\cos\left(\frac{n\sigma}{2}\right) \tag{9}$$

with $\alpha^r_{-\frac{n}{2}} = (\alpha^r_{\frac{n}{2}})^*$.