Exercise 1 – Closed strings on the orbifold $\mathbb{R}^1/\mathbb{Z}_2$

The purpose of this exercise is to define closed strings in a 26-dimensional space-time where the coordinate x^{25} is restricted to $x^{25} \geq 0$ by the identification

$$x^{25} \sim -x^{25} \ . \tag{1}$$

The string coordinate $X^{25}(\tau, \sigma)$ will be written as $X(\tau, \sigma)$, and the collection of string coordinates is therefore X^+, X^-, X^i and X, with $i = 2, \ldots, 24$. We introduce an operator U such that

$$UX(\tau, \sigma)U^{-1} = -X(\tau, \sigma) \ . \tag{2}$$

We also demand that $UX^iU^{-1} = X^i, Up^+U^{-1} = p^+$ and $Ux_0^-U^{-1} = x_0^-$. The *orbifold* closed string theory keeps only U-invariant states.

a) What is the U action on the modes x_0, p, α_n and $\tilde{\alpha}_n$ of the string coordinate X? What is the U action on X^-? Argue that the Hamiltonian is invariant, so that U is really a symmetry of the closed string theory.

b) Denote the original ground states as $|p^+, \vec{p}, p\rangle$, where $\vec{p}$ is a vector with components p^i and p denotes the momentum in the 25th direction. Assume that $U|p^+, \vec{p}, 0\rangle = |p^+, \vec{p}, 0\rangle$. What is $U|p^+, \vec{p}, p\rangle$? What are the ground states of the orbifold theory?

c) List the massless states of the orbifold theory.

It turns out that this is not the end of the story. The orbifold theory has a *twisted* sector which includes new kinds of closed strings. These closed strings can be imagined as open strings in the original space, but with the condition that the endpoints are identified by (1), i.e.

$$X(\tau, \sigma + 2\pi) = -X(\tau, \sigma) \ . \tag{3}$$

In this sector, the mode expansion for $X(\tau, \sigma)$ is different. As usual, we write

$$X(\tau, \sigma) = X_L(u) + X_R(v) \ , \quad u = \tau + \sigma \ , \quad v = \tau - \sigma \ . \tag{4}$$

d) Use (3) and (4) to develop a mode expansion for $X(\tau, \sigma)$. You may start by finding out how $X_L'(u)$ and $X_R'(v)$ behave when u and v are increased by 2π. You should find the final result

$$X(\tau, \sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}_{\text{odd}}} \frac{2}{n} e^{-i\frac{n}{2}\tau} \left(\alpha_{\frac{n}{2}} e^{i\frac{n}{2}\sigma} + \tilde{\alpha}_{\frac{n}{2}} e^{-i\frac{n}{2}\sigma} \right) \ . \tag{5}$$

The commutation relations for the modes can be obtained in a similar way as was done in exercise 3 of sheet 3. The result (which does not have to be derived) is

$$\left[\alpha_{\frac{n}{2}}, \alpha_{\frac{m}{2}}\right] = \frac{n}{2}\delta_{m+n,0} = \left[\tilde{\alpha}_{\frac{n}{2}}, \tilde{\alpha}_{\frac{m}{2}}\right] \ , \quad \left[\alpha_{\frac{n}{2}}, \tilde{\alpha}_{\frac{m}{2}}\right] = 0 \ . \tag{6}$$

e) Equation (2) must also hold in the twisted sector. What is the U action on your new oscillators? Describe the ground states in the twisted sector.

f) Use the zeta function regularization discussed in class to find the mass-squared formula in the twisted sector. Give the masses of the ground states. Exhibit the states at the first excited level and give their masses. Where do the fields in the twisted sector live?

Exercise 2 – Quantum Virasoro algebra

In exercise 3 of sheet 3, you derived the classical Virasoro algebra. In the quantum theory, it gets modified. One can verify that the ordering ambiguity of the quantum theory only affects the terms with $m+n=0$. Thus, the algebra takes the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0} \ . \tag{7}$$

a) Show that if $A(1) \neq 0$, it is possible to change the definition of L_0, by adding a constant, so that $A(1) = 0$.

b) Show that $A(-m) = -A(m)$.

c) Consider the Jacobi identity

$$[[L_m, L_n], L_k] + [[L_k, L_m], L_n] + [[L_n, L_k], L_m] = 0 \tag{8}$$

to show that for $m+n+k=0$ one has

$$(m-n)A(k) + (n-k)A(m) + (k-m)A(n) = 0 \ . \tag{9}$$

d) Consider equation (9) for $k=1$. Show that $A(1)$ and $A(2)$ determine all $A(n)$.

e) Assuming $A(1) = 0$ (which can always be reached by part a)), show that $A(m) = (m^3 - m)A(2)/6$.

f) Determine $A(2)$ by considering the matrix elements $\langle 0; 0|[L_1, L_{-1}]|0; 0\rangle$ and $\langle 0; 0|[L_2, L_{-2}]|0; 0\rangle$.