<u>Note:</u> **The 1st exam will take place on December 17, 8.15 am, in room A010 during the tutorial. The result (and also the solutions) will be available on the net shortly after:** *http://homepages.physik.uni-muenchen.de/~Michael.Haack//ST1_WS09.html* **. There will be no class on December 21. The first class next year will be on January 11, the first tutorial on January 14.**

Exercise 1 – Polarization tensor

Consider the closed string state

$$|\phi\rangle = \xi_{\mu\nu}\, \alpha^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1}\, |0;k\rangle \ , \tag{1}$$

where $\xi_{\mu\nu}$ denotes the polarization tensor. What do the Virasoro constraints on physical states,

$$L_m|\phi\rangle = \tilde{L}_m|\phi\rangle = 0 \quad , \quad m > 0 \ , \tag{2}$$

imply for $\xi_{\mu\nu}$?

Exercise 2 – Weights of some typical operators of the free X^μ CFT

a) By computing the OPEs with $T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu:$ and $\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu:$, confirm that the following operators have the indicated weights $(h, \tilde{h})$:

$$X^\mu \to (0,0) \ , \quad \partial X^\mu \to (1,0) \ , \quad \bar{\partial} X^\mu \to (0,1) \ , \quad \partial^2 X^\mu \to (2,0) \ , \quad :e^{ik\cdot X}: \to \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right) \ . \tag{3}$$

Which of the operators are tensor operators?

b) Calculate the singular terms in the OPE of $T(z_1)T(z_2)$.

Exercise 3 – The OPE of the energy momentum tensor

In the complex plane, the Virasoro generators L_n are given by

$$L_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} T(z). \tag{4}$$

a) Show that

$$[L_n, L_m] = \oint_{C_0} \frac{dz_2}{2\pi i} \oint_{C_{z_2}} \frac{dz_1}{2\pi i} z_1^{n+1} z_2^{m+1} T(z_1)T(z_2). \tag{5}$$

Here, C_0 denotes a contour about $z_2 = 0$, C_{z_2} is a contour about $z_1 = z_2$, and, as usual, the product $T(z_1)T(z_2)$ is meant to be the radially ordered product:

$$T(z_1)T(z_2) \equiv R(T(z_1)T(z_2)) = \left\{ \begin{array}{ll} T(z_1)T(z_2) & \text{for } |z_1| > |z_2| \\ T(z_2)T(z_1) & \text{for } |z_2| > |z_1| \end{array} \right\} . \tag{6}$$

(*Hint*: Write the commutator as a difference of two double contour integrals and use a contour deformation of the dz_1 integration for fixed z_2, just as was done in class for $\delta_Q \phi(\tilde{z}) = \epsilon[Q, \phi(\tilde{z})]$.)

b) Use (5) and the (radially ordered) operator product

$$T(z_1)T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial_{z_2} T(z_2)}{(z_1 - z_2)} + \text{ (finite terms)} \ , \tag{7}$$

as well as the Cauchy-Riemann formula,

$$\oint_{C_{z_2}} \frac{dz_1}{2\pi i} \frac{f(z_1)}{(z_1 - z_2)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z_2) \ , \tag{8}$$

to rederive the quantum Virasoro algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{m+n,0} \ . \tag{9}$$

c) Use (7) and the general formula for infinitesimal conformal transformations

$$\delta_\xi \phi(z) = \epsilon [L_\xi, \phi(z)] \ , \tag{10}$$

where

$$L_\xi = -\oint_{C_0} \frac{dz}{2\pi i} \xi(z) T(z) \ , \tag{11}$$

to show that

$$\delta_\xi T(z) = -\epsilon \Big[\frac{c}{12} \partial^3 \xi(z) + 2\partial\xi(z) T(z) + \xi(z) \partial T(z) \Big] \ . \tag{12}$$

What does this imply for $T(z)$?