Exercise 1 – Closed string vertex operator

Consider the operator

$$\zeta_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} : , \tag{1}$$

where $\zeta_{\mu\nu}$ is a constant tensor. Determine the condition for this operator to be primary, by looking at the OPE with the energy momentum tensor $T = (-1/\alpha') : \partial X_\rho \partial X^\rho :$. What is its weight?

Exercise 2 – Unitary CFT's

Show that a unitary CFT (i.e. one without negative norm states) has the following two characteristics:

(i) $c \geq 0$, where c is the central charge,

(ii) $h \geq 0$, where h is the weight of any primary field (and similar for $\tilde{h}$).

Hint: You need to make use of the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$.

Exercise 3 – Determinants and Graßmann numbers

Determinants of operators can formally be written as (path) integrals over a new set of auxiliary variables. In order for this to be possible, these auxiliary variables have to be anti-commuting rather than ordinary commuting numbers. Two anti-commuting numbers (or Graßmann numbers) θ and η satisfy

$$\theta\eta = -\eta\theta \tag{2}$$

and hence $\theta^2 = 0$. Because of this, the most general function of one Graßmann variable θ is

$$f(\theta) = A + B\theta \tag{3}$$

with $A, B \in \mathbb{C}$. Integrals over Graßmann variables ("Berezin integrals") are defined by

$$\int d\theta [A + B\theta] := B. \tag{4}$$

a) Defining the derivative

$$\frac{d}{d\theta}\theta = 1 , \qquad \frac{d}{d\theta}A = 0 , \qquad (A \in \mathbb{C}) , \tag{5}$$

show that the Berezin integral of a total derivative is zero and that the Berezin integral is translation invariant, i.e.,

$$\int d\theta \frac{d}{d\theta} f(\theta) \;=\; 0 , \tag{6}$$

$$\int d\theta f(\theta + a) \;=\; \int d\theta f(\theta) , \qquad \text{for } a \in \mathbb{C} . \tag{7}$$

These properties mimic similar properties of ordinary integrals of the type $\int_{-\infty}^{\infty} dx f(x)$, which is the motivation for the unusual definition (4). Note that, for Graßmann variables, integration and differentiation are equivalent operations.

b) If one has several linearly independent Graßmann variables θ_i $(i = 1, \ldots, n)$, where

$$\forall_{i,j}: \quad \theta_i \theta_j = -\theta_j \theta_i \ , \tag{8}$$

one defines

$$\int d\theta_1 \ldots d\theta_n f(\theta_i) = c \ , \tag{9}$$

where c is the coefficient in front of the $\theta_n \theta_{n-1} \ldots \theta_1$-term in $f(\theta_i)$ (note the order):

$$f = \ldots + c\theta_n \theta_{n-1} \ldots \theta_1 \ . \tag{10}$$

Let n be even and split the θ_i into two sets ψ_m, χ_m $(m = 1, \ldots, \frac{n}{2})$:

$$(\theta_1, \ldots, \theta_n) = (\psi_1, \chi_1, \psi_2, \chi_2, \ldots, \psi_{\frac{n}{2}}, \chi_{\frac{n}{2}}) \ . \tag{11}$$

Show that

$$\int \Big(\prod_{m=1}^{\frac{n}{2}} d\psi_m d\chi_m \Big) e^{\sum_{k=1}^{\frac{n}{2}} \chi_k \lambda_k \psi_k} = \prod_{m=1}^{\frac{n}{2}} \lambda_m \ , \tag{12}$$

where $\lambda_m \in \mathbb{C}$ are ordinary c-numbers and the exponential is defined via its power series expansion. Moreover, show that this implies

$$\int \Big(\prod_{m=1}^{\frac{n}{2}} d\psi_m d\chi_m \Big) e^{\sum_{k,l=1}^{\frac{n}{2}} \chi_k \Lambda_{kl} \psi_l} = \det \Lambda \ , \tag{13}$$

for a symmetric $\frac{n}{2} \times \frac{n}{2}$ matrix Λ with eigenvalues λ_m (this can be easily generalized to complex Graßmann numbers and Hermitian matrices).

c) Compare (12) with the result of the integral over commuting numbers α_m, β_m $(m = 1, \ldots, n/2)$, with $\lambda_m \in \mathbb{R}$. More concretely, show

$$\int_{-\infty}^{\infty} \Big(\prod_{m=1}^{\frac{n}{2}} d\alpha_m d\beta_m \Big) e^{2\pi i \sum_{k=1}^{\frac{n}{2}} \alpha_k \lambda_k \beta_k} = \prod_{m=1}^{\frac{n}{2}} \frac{1}{\lambda_m} \ . \tag{14}$$

<u>*Comments:*</u> The fact that one can invert the result of a Gaussian integral by replacing the commuting variables by Graßmann valued variables, carries over to path integrals. This is commonly used in QFT where fermionic path integrals are used to express the determinant of a differential operator. For instance, formula (13) can be generalized to the context of a path integral over Graßman valued fields $\psi(x)$, $\chi(x)$, resulting in

$$\int \mathcal{D}\psi \mathcal{D}\chi \ e^{\int d^D x \chi \Delta \psi} = \det \Delta \ , \tag{15}$$

where Δ is some self-adjoint differential operator. This can be seen as follows. The fields $\psi(x)$ and $\chi(x)$ can be expanded in (c-number valued) eigenfunctions $\Psi_i(x)$ of Δ with Graßmann valued coefficients ψ_i and χ_i, i.e.

$$\psi(x) = \sum_i \psi_i \Psi_i(x) \quad , \qquad \chi(x) = \sum_i \chi_i \Psi_i(x) \ ,$$
$$\Delta \Psi_i(x) \ = \ \lambda_i \Psi_i(x) \ . \tag{16}$$

The eigenfunctions can be chosen in an orthonormal way, i.e.

$$\int d^D x \Psi_i(x) \Psi_j(x) = \delta_{ij} \ , \tag{17}$$

and the measure can be defined as $\mathcal{D}\psi \mathcal{D}\chi = \prod_i d\psi_i d\chi_i$.