① Higher Curvature Corrections From Loop Cosm
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AIM: Understand what is going on in LQG with a simple example

- uniqueness
- finiteness
- "new quantum mechanics"

PRELUDE: Naive perturbative quantization of gravity

write $g_{\mu\nu} = \eta_{\mu\nu} + \varkappa h_{\mu\nu}$, gauge fixing

$$\int \sqrt{g} R = \int h \Box h + \varkappa h^3 + \varkappa^2 h^4 + \dots$$

loop diagrams increasingly divergent.

⇝ regularize e.g. $\int^{1/l_p} d^4p \dots$ or $\int dx \int dy \dots$ or ...
$|x-y| > l_p$

cut-off point splitting

⇝ effective action

$$\int \sqrt{-g} \left(\# R + \# \frac{1}{l_p^2} R^2 + \# \frac{1}{l_p^2} \Box R + \# \frac{1}{l_p^2} R_{\mu\nu} R^{\mu\nu} + \# R^3 + R^4 \dots \right.$$
$$\left. (+ \text{non-covariant terms}) \right)$$

are finite numbers that strongly depend on regularization.

⇝ We could have started already from a higher derivative theory and would have obtained similar but different coefficients.

The regularized theory is finite and the coefficients can be obtained in a deterministic manner

BUT: arbitrary, not predictive

Restrict to cosmological case:

- homogeneous
- isotropic
- add a scalar clock field (as t is invariant under shifts)

$$ds^2 = -dt^2 + a(t)^2 (d\vec{x}^2)$$

$$\phi = \phi(t)$$

$$S = \int d^4x \sqrt{-g}\,(R - g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi) = \cancel{\int dt\,(-6a\dot{a}^2 + a^3\dot{\phi}^2)}$$

$$= \int dt\, a^3 \left(-\left(\frac{\dot{a}}{a}\right)^2 + \dot{\phi}^2\right)$$

loopy story suggests change of variables (volume will later have discrete spectrum)

$$V = a^3$$

$$\rightsquigarrow L = -\frac{2}{3}\frac{\dot{V}^2}{V} + V\dot{\phi}^2$$

NB: Do not use Euler-Lagrange Equations (2nd order!) as the system is still invariant under $t \mapsto t+c$. Rather impose constraint $H \approx 0$.

In terms of canonical momenta

$$\beta = \pi_v = -\frac{4}{3}\frac{\dot{v}}{v} \qquad p = \pi_\phi = 2v\dot{\phi}$$

$$H = -\frac{3}{8}v\beta^2 + \frac{1}{4}\frac{p^2}{v} = \frac{1}{v}\left(\sqrt{\tfrac{3}{8}}\,v\beta + \frac{p}{2}\right)\left(-\sqrt{\tfrac{3}{8}}\,v\beta + \frac{p}{v}\right)$$

so constraint reads $\sqrt{\frac{3}{8}}\,v\beta \approx \pm\frac{p}{2}$.

~~use ϕ as a clock variable~~

eom: $\dot{p} = 0$, $\dot{\phi} = \frac{p}{2v}$, $\dot{v} = -\frac{3}{v}\beta \approx \pm\sqrt{\frac{3}{8}}\,p$

$\rightsquigarrow$ $v = \left|\sqrt{\frac{3}{8}}\,p\,(t-t_0)\right|$, $\phi = \sqrt{\frac{2}{3}}\ln|t-t_0|$

in terms of clock $\frac{dv}{d\phi} = \frac{\partial H}{\partial \beta}\Big/\frac{\partial H}{\partial p} = \frac{3\beta v^2}{2p} \approx \pm\sqrt{\frac{3}{4}}\,v \rightsquigarrow v = e^{\pm\sqrt{\frac{3}{4}}(\phi-\phi_0)}$

ie $v = 0$ for $t = t_0$ $\rightsquigarrow$ Big Bang singularity

This is the classical story

In loop cosmology one would like to use canonical quantisation. Due to an unfortunate choice of (Polymer) Hilbert space, however, ~~β cannot be~~ $\hat{\beta} = -i\frac{\partial}{\partial v}$ is too singular to be an operator.

But a translation $(U(a)\psi)(v) = \psi(v-a) = (e^{i\beta a}\psi)(v)$ is well defined (and unitary).

So one can regularize

$$\tilde{\beta} = \frac{U(\ell) - U(-\ell)}{2i\ell} = \sin \ell\beta/\ell$$

without taking the singular $\ell \to 0$ limit.

in the full 3+1-dimensional theory where

$$ds^2 = -N^2 dt^2 + g^{(3)}_{ij}\,(dx^i + N^i)(dx^j + N^j)$$

this corresponds to $R^{(3)}$ not being a well defined operator.

There the solution is to express it in terms of holonomies (which do exist) around small loops $\square_\ell^{e}$ without taking $\ell \to 0$.

We can proceed at the classical level write

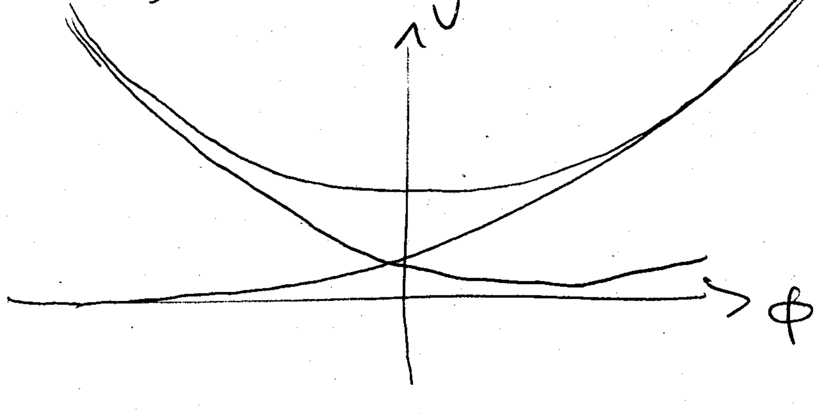
$$\tilde{H} = -\frac{3}{8}\,\frac{v \sin(\ell\beta)^2}{\ell^2} + \frac{p^2}{4v} = \frac{1}{v}\left(\sqrt{\frac{3}{8}}\,\frac{v\sin\ell\beta}{\ell} + \frac{p}{2}\right)\left(-\sqrt{\frac{3}{8}}\,\frac{v\sin\ell\beta}{\ell} + \frac{p}{2}\right)$$

As above

$$\frac{dv}{d\phi} = \frac{\partial H/\partial\beta}{\partial H/\partial p} = \sqrt{\frac{3}{2}}\, v\cos\ell\beta \simeq \pm\sqrt{\frac{3}{2}v^2 - (\ell p)^2}$$

$$\rightsquigarrow \quad v = \sqrt{\frac{3}{2}}\,\ell|p|\cosh\left(\sqrt{\frac{3}{2}}(\phi - \phi_0)\right)$$

$\rightsquigarrow$ bounce avoids singularity.



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<u>Ambiguities</u> (Counterterms)

$\dfrac{U(\ell)-U(-\ell)}{2\ell}$ is not the only expression in $U(\ell)$ with $\lim_{\ell\to 0} \ldots = \dfrac{\partial}{\partial v_0}$

e.g. two such expressions

$$\hat{\beta}_\alpha = \alpha\,\frac{U(\ell)-U(-\ell)}{2i\ell} + (1-\alpha)\,\frac{U(2\ell)-U(-2\ell)}{4i\ell}$$

$$= \beta + \frac{3\alpha-4}{6}\,\ell^2\beta^3 + O(\ell^4\beta^5)$$

or many more.

We only need $\tilde{\beta} = \dfrac{f(\ell\beta)}{\ell}$

with $f(0)=0$, $f'(0)=1$. Any such function works ("improved lattice actions").

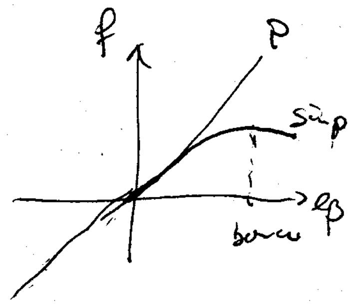
"One function worth of ambiguities"

In general, we have

$$H_f = -\frac{3}{8}\,\frac{v\,f(\ell\beta)^2}{\ell^2} + \frac{p^2}{4v} = \frac{1}{v}\left(\sqrt{\frac{3}{8}}\,\frac{v f(\ell\beta)}{\ell} + \frac{p}{2}\right)\left(-\sqrt{\frac{3}{8}}\,\frac{v f(\ell\beta)}{\ell} + \frac{p}{2}\right)$$

$$\rightsquigarrow \quad \frac{dv}{d\phi} = \sqrt{\frac{3}{2}}\; v\, f'(\ell\beta)$$

Bounce $\hat{=}$ $\dfrac{dv}{d\phi} = 0$ ie $f'(\ell\beta) = 0$



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Conclusion: We regularize by adding terms with higher powers of $\beta = \pi_v$ to the Hamiltonian.

Retranslate to Lagrangian via

$$L = \dot{v}\beta + \dot{\phi}p - H$$

$\rightsquigarrow$ solve $\dot{v} = \frac{\partial H}{\partial \beta} = -\frac{3}{4} v f(\ell\beta) f'(\ell\beta)/\ell$ for β.

in LQC case $f = \sin$

$$\tilde{L} = \frac{3}{16}\frac{a^3}{\ell^2}\left(1 - \sqrt{1 - H^2} - H \arcsin H\right) + a^3 \dot{\phi}^2$$

w/ $H = 8\ell \frac{\dot{a}}{a}$ Hubble "constant"

$$\lim_{\ell \to 0} \tilde{L} = L_{orig}$$

but for finite $\ell > 0$ arbitrary high powers of h
$\hat{=}$ higher curvature corrections

Other choices f give other corrections.

Fz

To see this: Recovariantize L.

Technical complication

$$R = 6\left(\frac{\dot a}{a}\right)^2 - 6\,\frac{\ddot a}{a}$$

for $\int \sqrt{g}\, R$ can be integrated by parts to $\left(\frac{\dot a}{a}\right)^2$ - term.

But: Non-trivial $f(R)$ - gravity has $L(a, \dot a, \ddot a)$ and 3^{rd} order e.o.m. or additional degree of freedom
$\rightsquigarrow$ Hamilton picture unclear.

Way out:

Consider products of traces of Ricci - <u>tensor</u>

$$tr(R_\cdot^{\cdot\, N}) = R_{\mu_1}{}^{\mu_2} R_{\mu_2}{}^{\mu_3} R_{\mu_3}{}^{\mu_4} \cdots R_{\mu_N}{}^{\mu_1}$$

paper: Any polynomial $a^3\, P\left(\frac{\dot a}{a}\right)$ can be written in such a way (up to integration by parts)

$\rightsquigarrow$ all f - choices can be written as higher curvature actions

Covariantization?

Conclusions:

- LQC situation is similar to naive perturbative treatment
  - regularization via choice of f
  - this induces counter terms that could have been added in the beginning
  - Predictiveness? effective action
- Likely to extend to full 4D LQG
  - that might even fail to be Lorentz covariant
- Higher curvature actions possible which still have 1st order FRW-type equations.

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Why the polymer Hilbert space?

Caricature of a derivation
(~~.....~~ but spirit of argument is preserved)

You want to do quantum mechanics of x and p
with $[x,p] = i$.
This leads to unbounded operators.
Better choice (true!): Weyl operators

$$U(a) = e^{iax} \quad , \quad V(b) = e^{ibp}$$

with $U(a)\, V(b) = V(b)\, U(a)\, e^{iab}$ (BCH)

Hilbert space: functions $\psi : \mathbb{R} \to \mathbb{C}$.

BUT: Before introducing Hamiltonian, the problem is translationally invariant under $x \mapsto x + c$

$$(U(a)\psi)(x) = e^{iax}\psi(x) \, , \quad (V(b)\psi)(x) = \psi(x+b) \quad \text{does the job.}$$

So, there should also be an invariant state
("ground-state"/ "vacuum"/ GNS state)
$\rightsquigarrow$ require $(\psi : x \mapsto 1) \in \mathcal{H}$.

By applying U's and V's:
All $\psi(x) = e^{ikx}$ are in $\mathcal{H}$.

+ linear combinations (in L^2 sense).

- Normalization

$$\langle e | e \rangle = 1$$

$U(a)$ unitary $\Rightarrow$ $\langle e^{icx}, e^{icx} \rangle = 1$

$V(b)$ unitary

$$\langle e^{icx}, e^{idx} \rangle \overset{V(b)}{=} \langle e^{ibc} e^{icx}, e^{ibd} e^{idx} \rangle$$

$$= e^{i(d-c)b} \langle e^{icx}, e^{idx} \rangle$$

$\Rightarrow$ $d = c$ $\vee$ $\langle e^{icx}, e^{idx} \rangle = 0$

all planewaves are orthogonal..

Infinitessimal operators

$$p = \lim_{\varepsilon \to 0} \frac{V(\varepsilon) - V(-\varepsilon)}{2\varepsilon}$$ does exist:

$$p \, e^{ikx} = ik \, e^{ikx}.$$

It even has eigenvectors!

$$x \text{ "} = \lim_{\varepsilon \to 0} \frac{U(\varepsilon) - U(-\varepsilon)}{2\varepsilon} \text{"}$$ does not exist

since $U(\varepsilon) e^{ikx} \perp U(-\varepsilon) e^{ikx}$ $\forall \varepsilon > 0$

In the cosmological setting: $\beta = x$, $v = p$.