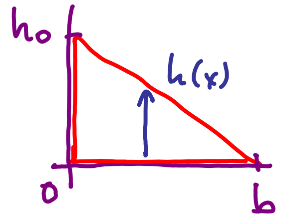
Flächenintegral: Integrationsgrenzen definieren die Fläche:

07.-22.11.05 F32a

Beispiele: Fläche v. Dreieck:

$$h(x) = h_0(1 - x/b)$$



$$A = \int dA = \int_0^b dx \int_0^{h(x)} dy = \int_0^b dx\, h(x) = \int_0^b dx\, h_0(1 - x/b) = \qquad (1)$$

$$= h_0\left(b - \tfrac{1}{2}\, b^2/b\right) = \tfrac{1}{2} h_0 b$$

Dichte (pro Fläche) des Dreiecks sei gegeben durch: $\rho(x,y) = x^2 y$.

Was ist Masse des Dreiecks?

$$M = \int dA\, \rho(x,y) = \int_0^b dx \int_0^{h(1-x/b)} dy\; x^2 y = \int_0^b dx\, x^2\, \tfrac{1}{2} y^2 \Big|_0^{h(1-x/b)} \qquad (2)$$

$$(3) \qquad = \int_0^b dx\, x^2\, \frac{h_0^2}{2}(1 - x/b)^2 = \frac{h_0^2}{2}\int_0^b dx\, x^2\left[1 - \frac{2x}{b} + \frac{x^2}{b^2}\right]$$

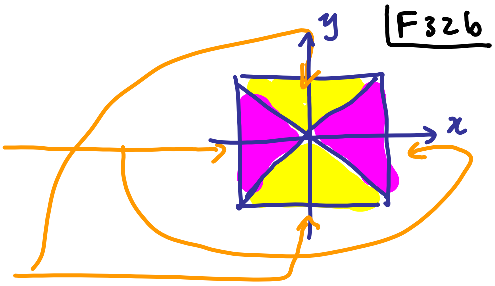
$$(4) \qquad = \frac{h_0^2}{2}\left[\frac{1}{3}b^3 - \frac{2}{b}\frac{1}{4}b^4 + \frac{1}{b^2}\frac{1}{5}b^5\right] = h_0^2 b^3\, \underbrace{\tfrac{1}{2}\left[\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5}\right]}_{1/60}$$

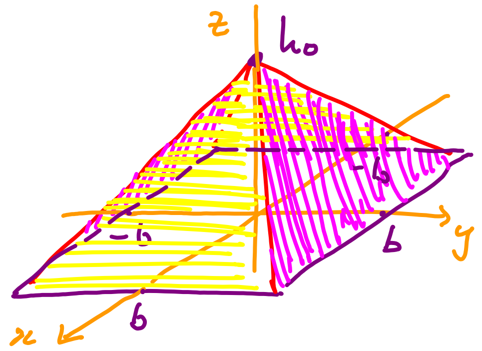
Volumenintegral: Volumen einer Pyramide

F32b

Höhe:

$$h(x,y) = \begin{cases} h_0\left(1 - \frac{|x|}{b}\right) & \forall\ |x| > |y| \\ h_0\left(1 - \frac{|y|}{b}\right) & \forall\ |x| < |y| \end{cases}$$





$$\text{Volumen} = \int dV = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_0^{h(x,y)} dz$$

Integriere getrennt über die Bereiche $|x| > |y|$ und $|x| < |y|$:

$$\text{Volumen} = \int_{-b}^{b} dx \int_{-|x|}^{|x|} dy \int_0^{h_0(1-|x|/b)} dz + \int_{-b}^{b} dy \int_{-|y|}^{|y|} dx \int_0^{h_0(1-\frac{|y|}{b})} dz$$

liefert dasselbe

$$= \int_{-b}^{b} dx\; 2|x|\, h_0(1 - |x|/b) + \int_{-b}^{b} dy\; 2|y|\, h_0\left(1 - \frac{|y|}{b}\right)$$

$$= 2 \cdot 4 h_0 \int_0^b dx \left(x - \frac{x^2}{b}\right) = 8 h_0\left[\tfrac{1}{2} b^2 - \tfrac{1}{3}\, b^3/b\right] = \tfrac{4}{3} h_0 b^2$$

Dichte pro Volumen der Pyramide sei $\rho(x,y,z) = z$ F32c

Masse : $M = \int dV \rho(x,y,z) = 4 \int_0^b dx \int_0^b dy \int_0^{h(x,y)} dz \; z$

$$= \int_{-b}^{b} dx \int_{-|x|}^{|x|} dy \int_0^{h_0(1-|x|/b)} dz \; z \quad + \quad \int_{-b}^{b} dy \int_{-|y|}^{|y|} dx \int_0^{h_0(1-\frac{|y|}{b})} dz \; z$$

liefert dasselbe

$$= \int_{-b}^{b} dx \; 2|x| \; \tfrac{1}{2} h_0^2 \left(1 - |x|/b\right)^2 \quad + \quad \int_{-b}^{b} dy \; 2|y| \; \tfrac{1}{2} h_0 \left(1 - |y|/b\right)^2$$

$$= 2 \cdot 2 \cdot h_0^2 \int_0^b dx \left(x - 2x^2/b + x^3/b^2\right)$$

$$= 2 h_0^2 \left[\tfrac{1}{2} x - \tfrac{2}{3} x^3/b + \tfrac{1}{4} \frac{x^4}{b^2} \right]_0^b$$

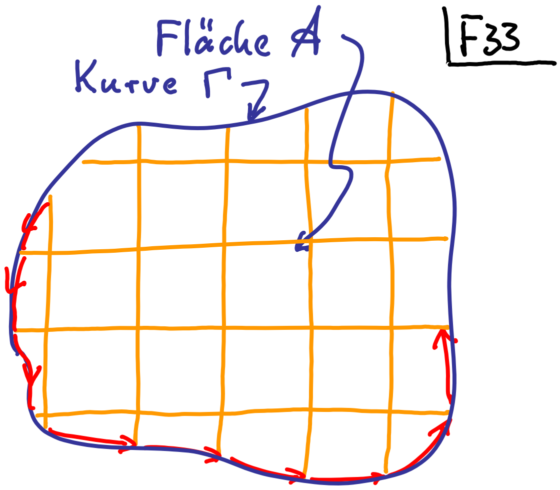
$$= 2 h_0^2 b \left[\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} \right] \qquad \frac{6-4+3}{12} = \frac{5}{12}$$

$$= \frac{5}{6} h_0^2 b$$

Stoke'sche Satz

07 - 21.11.05 F33

Betrachte endliche Fläche $\mathcal{A}$, umschlossen durch Kurve Γ:



Zirkulation um $\mathcal{A}$: $\quad Z_{\mathcal{A}} \overset{(30.1)}{\equiv}$ (1)

alternativ

$\sum$ (Zirkulation pro Flächenelement) (Flächen-element)

über alle Flächenelemente

Zirkulation um ein typisches Flächenelement (2)

$\xrightarrow{\lim \Delta A \to 0}$

Flächenintegral Linienintegral

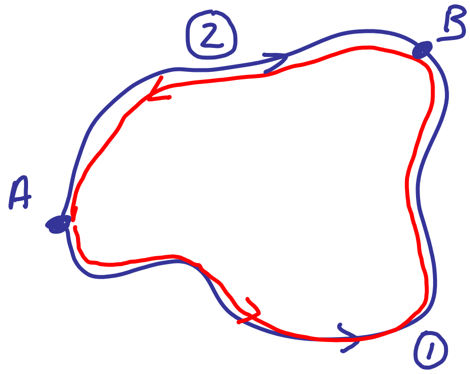
$$\boxed{\qquad \overset{(1)}{=} \qquad}$$ (3)

"Stoke'sche Satz"

gilt für "beliebige" Vektorfelder, beliebige $\mathcal{A}$, Γ

Noch ein Beweis für wegunabhängigkeit von $\int d\vec{r}\cdot\vec{\nabla}\varphi$

$$\int_{①: A\to B} d\vec{r}\cdot\vec{\nabla}\varphi \;-\; \int_{②: A\to B} d\vec{r}\cdot\vec{\nabla}\varphi$$



$$= \oint d\vec{r}\cdot\vec{\nabla}\varphi$$

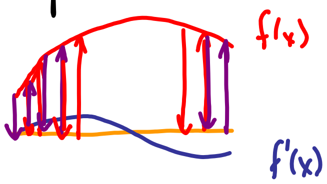
Stokes: $\overset{(33.3)}{=}$

Gradientenfelder sind rotationsfrei: $=$ (32.1)

$\Longrightarrow$ Weg 1 und Weg 2 liefern ✓

Zusammenfassung: 3 Integralsätze der Vektoranalysis

Hauptsatz der Analysis:

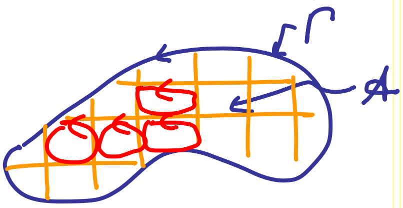


1-dim. Integral — 0-dim. Integral

$$\int_a^b dx\,\underbrace{(\partial_x f)}_{} = \lim_{\Delta x\to 0}\left(\frac{\Delta f}{\Delta x}\right)\Delta x = \underbrace{f(b) - f(a)}_{} \qquad (1)$$

Steigung = Änderung pro Abstand — Gesamtänderung

Stokes'sche Satz:

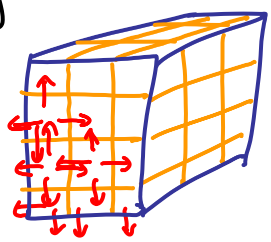


2-dim. Integral — 1-dim. Integral

$$\int_A d\vec{A}\cdot\underbrace{(\vec{\nabla}\times\vec{v})}_{} = \lim_{\Delta A\to 0}\sum \Delta A\left(\frac{\Delta Z}{\Delta A}\right) = \underbrace{\int_\Gamma d\vec{r}\cdot\vec{v}}_{} \qquad (2)$$

Zirkulation pro Flächenelement — Gesamtzirkulation

Gauss'sche Satz:



3-dim. Integral — 2-dim. Integral

$$\int_V dV\,(\vec{\nabla}\cdot\vec{v}) = \lim_{\Delta V\to 0}\sum \Delta V\left(\frac{\Delta\,\text{Fluss}}{\Delta V}\right) = \int_A d\vec{A}\cdot\vec{v} \qquad (3)$$

Ausfluss pro Volumenelement — Gesamtausfluss

Krummlinige Koordinatensysteme : Ebene Polarkoordinaten in $\mathbb{E}^2$

Transformationsregeln:

$$x = \rho\cos\varphi \quad (1a)$$
$$y = \rho\sin\varphi \quad (1b)$$

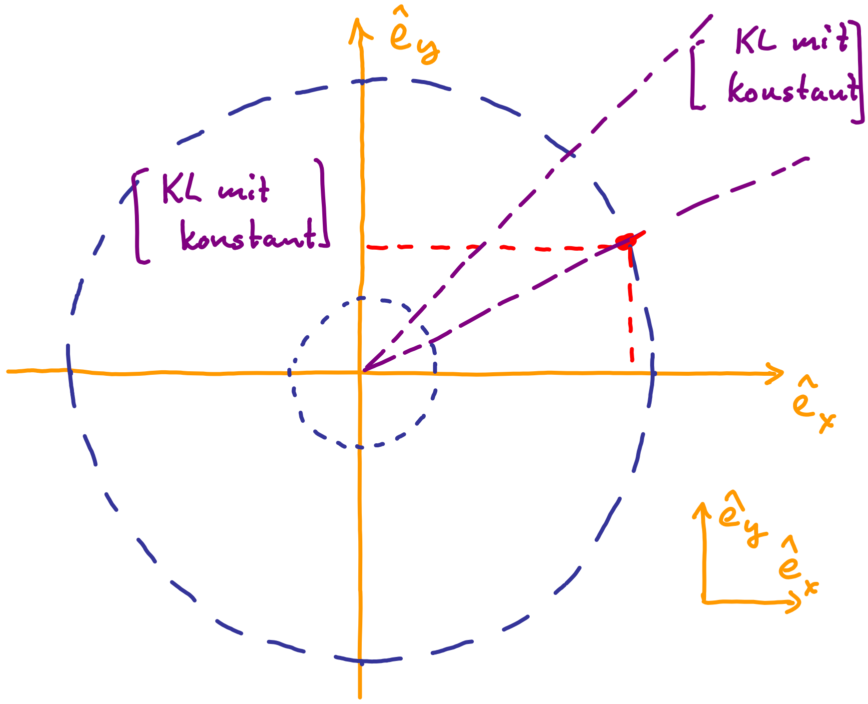
Umgekehrte Transformation:

$$\rho = \sqrt{x^2+y^2} \quad (2a)$$
$$\varphi = \arctan(y/x) \in [0,2\pi) \quad (2b)$$

$\vec{r} = \vec{r}(x,y)$ mit $\left\{ \begin{matrix} = \text{konst.} \\ = \text{konst.} \end{matrix} \right\}$

definiert "Koordinatenlinie" (KL)

parametrisiert durch { }



$\vec{r} = \vec{r}(\rho,\varphi)$ mit $\left\{ \begin{matrix} = \text{konst.} \\ = \text{konst.} \end{matrix} \right\}$

definiert "Koordinatenlinie" (KL)

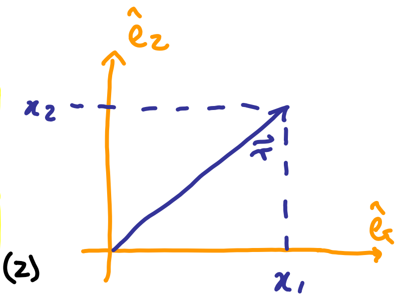
parametrisiert durch { }

Lokales Zweibein:

Ortsvektor in cartesischen Koord: $\vec{r}(x,y) =$ (1)

Lokales Zweibein, cartesisch: $\vec{r} \overset{(1)}{=} \sum_{i=1}^{2} x_i \hat{e}_i$, $\boxed{\begin{matrix} \dfrac{\partial \vec{r}(x_1,x_2)}{\partial x_i} = \\ = \text{parallel zur KL} \end{matrix}}$ (2)

$\Rightarrow$ Basisvektoren sind tangential zu KL



Dieselbe Konstruktion in Polarkoordinaten:

Ortsvektor [= (1)] in Polarkoordinaten: $\vec{r}(\rho,\varphi) \overset{(37.1)}{=} \underbrace{\qquad}_{x_1} \hat{e}_x + \underbrace{\qquad}_{x_2} \hat{e}_y$ (3)

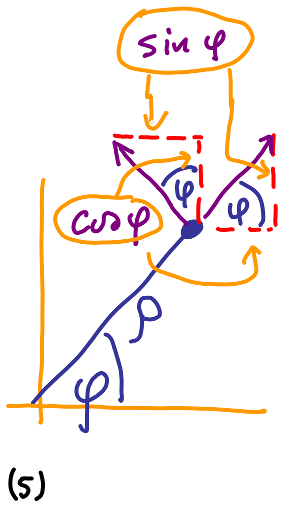
Lokales Zweibein, in Polarkoordinaten: [per Konstruktion tangential zu KL]

$$\frac{\partial \vec{r}(\rho,\varphi)}{\partial \rho} \overset{(3)}{=} \quad (4)$$

Betrag

$$\frac{\partial \vec{r}(\rho,\varphi)}{\partial \varphi} \overset{(3)}{=} \quad (5)$$

Betrag



Allgemein: $\hat{e}_\rho, \hat{e}_\varphi$ bilden ein lokales (ortsabhängiges) orthonormales Zweibein:

$$\hat{e}_\rho \equiv \frac{\partial \vec{r}}{\partial \rho} \Big/ \left| \frac{\partial \vec{r}}{\partial \rho} \right| = \cos\varphi\, \hat{e}_x + \sin \hat{e}_y \qquad (1)$$

$$\hat{e}_\varphi \equiv \frac{\partial \vec{r}}{\partial \varphi} \Big/ \left| \frac{\partial \vec{r}}{\partial \varphi} \right| = -\sin\varphi\, \hat{e}_x + \cos \hat{e}_y \qquad (2)$$

$$\vec{r} = x\hat{e}_x + y\hat{e}_y = \rho\cos\varphi\, \hat{e}_x + \rho\sin\varphi\, \hat{e}_y = \qquad (3)$$

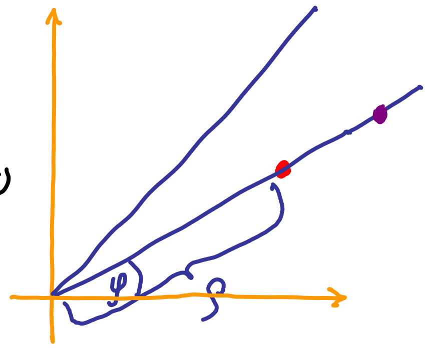
WICHTIG!!

Wegelement:

Skizze: siehe Seite 37.

Cartesisch: $d\vec{r} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy =$ (4)

polar: $= \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \varphi} d\varphi =$ (5)

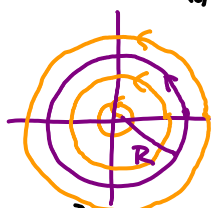


$\Rightarrow$ $$\boxed{d\vec{r} = d\rho\, \hat{e}_\rho + \rho\, d\varphi\, \hat{e}_\varphi} \qquad (6)$$

Dimensionen: Länge dimensionslos!

Flächenelement: $dA = dx\,dy =$ (7)

Bsp: Fläche eines Kreises: $A = \int dA = \int_0^r \rho\, d\rho \int_0^{2\pi} d\varphi = \frac{1}{2} r^2 2\pi = \pi r^2$ (1)



Bsp: $\vec{v}(\vec{r}) = r\,\hat{e}_\varphi$ sei Vektorfeld im 2-dimensionalen:

Überprüfen Sie Stoke'schen Satz (33.3) für den Kreisweg $\vec{r} = R$, $\varphi \in [0, 2\pi]$

$d\vec{r} = R\, d\varphi\, \hat{e}_\varphi$ (2)

$$\oint_{\text{Rand des Kreises}} d\vec{r} \cdot \vec{v} = \int_0^{2\pi} \overbrace{R\, d\varphi\, \hat{e}_\varphi}^{d\vec{r}} \cdot \overbrace{(R\,\hat{e}_\varphi)}^{\vec{v}(R)} = R^2 \int_0^{2\pi} d\varphi = 2\pi R^2$$

$\vec{\nabla} \times \vec{r} = ?$

$$\vec{v}(\vec{r}) \overset{(39.2)}{=} r(-\sin\varphi\, \hat{e}_x + \cos \hat{e}_y) = -y\,\hat{e}_x + x\,\hat{e}_y$$

$$\vec{\nabla} \times \vec{r} = (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)$$

$$= (0, 0, 2) = 2\hat{e}_z$$

$(\rho\, d\rho\, d\varphi\, \hat{e}_z)$

Stokes (33.3)

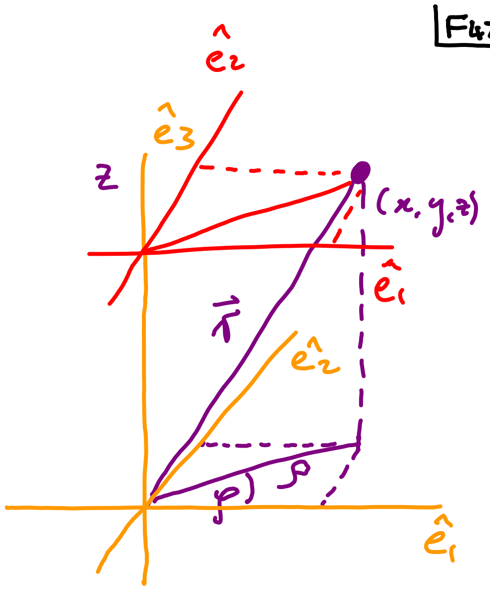
$$\int_{\text{Fläche des Kreises}} d\vec{A} \cdot (\vec{\nabla} \times \vec{v}) = \int_0^R \rho\, d\rho \int_0^{2\pi} d\varphi\, \underbrace{\hat{e}_z \cdot (\vec{\nabla} \times \vec{v})}_{2} = \left(\frac{1}{2} R^2\right) 2\pi \cdot 2 = 2\pi R^2$$

Polarkoordinaten in 3 Dimension

Def: $x = \rho\cos\varphi$, (1a) $\rho = \sqrt{x^2+y^2}$ (1d)

$y = \rho\sin\varphi$ (1b) $\varphi = \arctan(y/x) \in [0, 2\pi)$ (1e)

$z = z$ (1c) $z = z$ (1f)



Ortsvektor: $\vec{r} = \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z$ (2)

Polares Dreibein: (analog zu 2-D)

$$\hat{e}_\rho \overset{(2)}{\equiv} \frac{\partial\vec{r}}{\partial\rho}\Big/\left|\frac{\partial\vec{r}}{\partial\rho}\right| = \cos\varphi\,\hat{e}_x + \sin\,\hat{e}_y \quad (3a)$$

$$\hat{e}_\varphi \overset{(2)}{\equiv} \frac{\partial\vec{r}}{\partial\varphi}\Big/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = -\sin\varphi\,\hat{e}_x + \cos\,\hat{e}_y \quad (3b)$$

$$\hat{e}_z \overset{(2)}{\equiv} \frac{\partial\vec{r}}{\partial z}\Big/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = \hat{e}_z \quad (3c)$$

Wegelement: $d\vec{r} = d\rho\,\hat{e}_\rho + \rho\,d\varphi\,\hat{e}_\varphi$ (4)

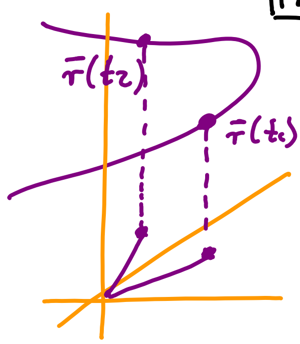
Volumenelement: $dV =$ (5)

Anwendung auf Bahnkurve:

zeitunabhängig

Ortsvektor: $\vec{r}(t) = x(t)\,\hat{e}_x + y(t)\,\hat{e}_y + z(t)\,\hat{e}_z$ (1)

zeitabhängig! $=$ (2)



(3)

Produktregel

Geschwindigkeitsvektor: $\vec{v}(t) = \frac{d\vec{r}}{dt} \overset{(2)}{=}$

Berechne $\dot{\hat{e}}_\rho$, $\dot{\hat{e}}_\rho$ Cartesischen Koord., wo Basisvekt. zeit unabhängig sind:

$$\dot{\hat{e}}_\rho \overset{(39.1)}{=} \frac{d}{dt}\left[\cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y\right]$$

$$\dot{\hat{e}}_\varphi \overset{(39.2)}{=} \frac{d}{dt}\left[-\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y\right]$$

$$\Rightarrow \quad \vec{v}(t) = \dot\rho\,\hat{e}_\rho + \rho \qquad + \dot z\,\hat{e}_z$$

Beschleunigung: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot\rho\,\hat{e}_\rho + \dot\rho\,\underbrace{\dot{\hat{e}}_\rho}_{\dot\varphi\hat{e}_\varphi} + \dot\rho\,\dot\varphi\,\hat{e}_\varphi + \rho\,\ddot\varphi\,\hat{e}_\varphi + \rho\,\dot\varphi\,\underbrace{\dot{\hat{e}}_\varphi}_{-\dot\varphi\hat{e}_\rho} + \ddot z\,\hat{e}_z$

$$= (\ddot\rho - \rho\dot\varphi^2)\,\hat{e}_\rho + (\rho\ddot\varphi + 2\dot\rho\dot\varphi)\,\hat{e}_\varphi + \ddot z\,\hat{e}_z$$

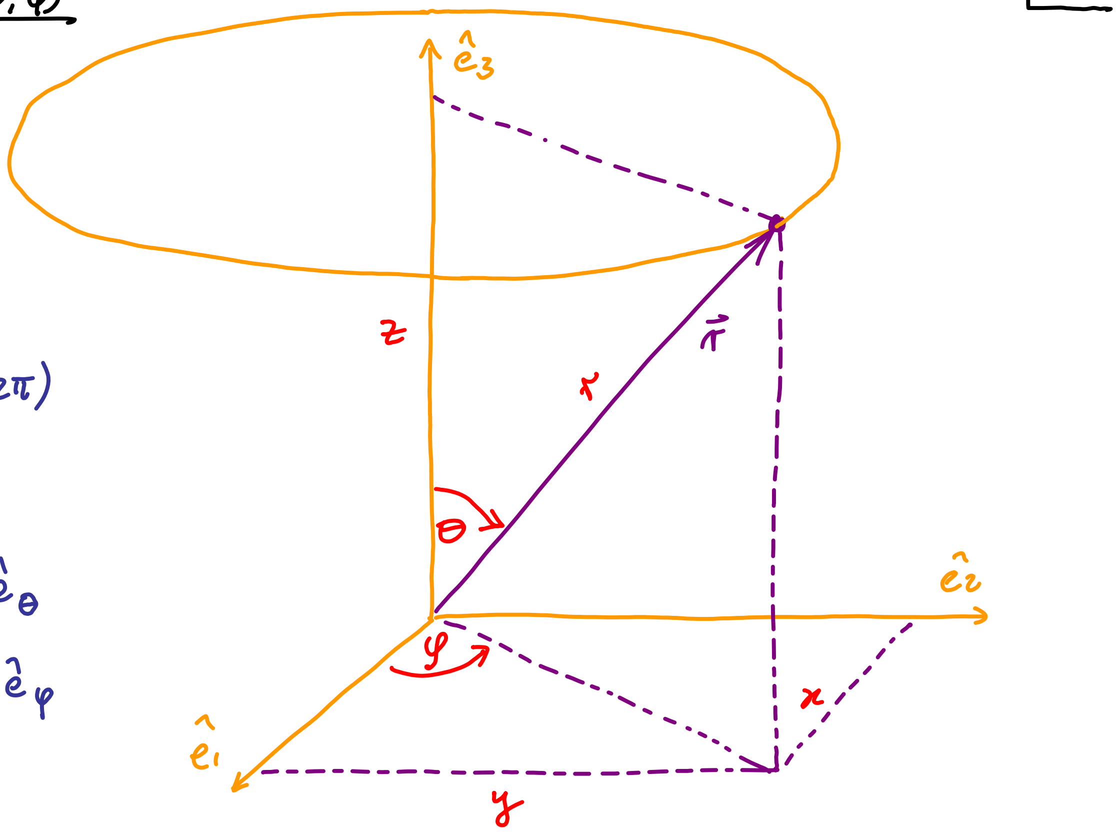
Kugelkoordinaten : (r, θ, φ)

$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

$$\theta \in [0,\pi], \quad \varphi \in [0, 2\pi)$$



Wegelement :

$$d\vec{r} = dr\, \hat{e}_r + d\theta\, r\, \hat{e}_\theta + d\varphi\, r \sin\theta\, \hat{e}_\varphi$$

Volumenelement :

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Allgemeine Koordinatentransformationen in 3D

Def. der Transf: $x_i = x_i(\quad)$, $i = 1,2,3$; $j = 1,2,3$ (1)

Cartesisch krummlinig

Ortsvektor: $\vec{r} =$ (2)

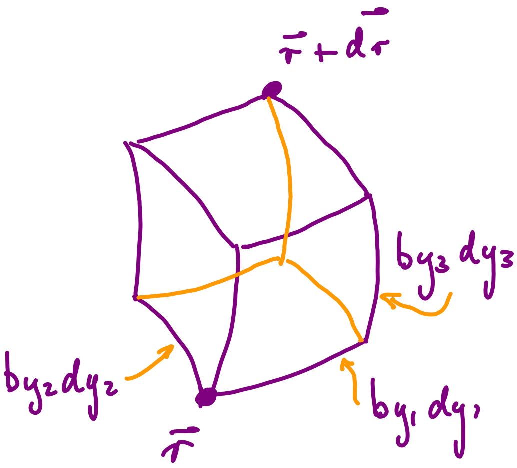
Normierungsfaktor

Lokales Dreibein: $\hat{e}_{y_i} \equiv$, mit $b_{y_i} \equiv$ $i = 1,2,3$ (3)

Wegelement: $d\vec{r} =$ (4)

$\overset{(3)}{=}$ (5)

Dimension : Länge



Volumenelement: $dV = dx_1 dx_2 dx_3$

$=$ (6)

Beispiel Kugelkoordinaten: F45

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta \tag{1}$$

$$\vec{r} = \sum_i x_i \hat{e}_i = r\left[\sin\theta\cos\varphi\,\hat{e}_1 + \sin\theta\sin\varphi\,\hat{e}_2 + \cos\theta\,\hat{e}_3\right] \tag{2}$$

$$\frac{\partial\vec{r}}{\partial r} \overset{(2)}{=} \left[\sin\theta\cos\varphi\,\hat{e}_1 + \sin\theta\sin\varphi\,\hat{e}_2 + \cos\theta\,\hat{e}_3\right] \tag{3}$$

$$\frac{\partial\vec{r}}{\partial\theta} \overset{(2)}{=} r\left[\cos\theta\cos\varphi\,\hat{e}_1 + \cos\theta\sin\varphi\,\hat{e}_2 - \sin\theta\,\hat{e}_3\right] \tag{4}$$

$$\frac{\partial\vec{r}}{\partial\varphi} \overset{(2)}{=} r\left[-\sin\theta\sin\varphi\,\hat{e}_1 + \sin\theta\cos\varphi\,\hat{e}_2\right] \tag{5}$$

$$\text{mit} \quad b_r = \quad , \quad b_\theta = \quad \quad b_\varphi = \tag{6}$$

$$d\vec{r} \overset{(44.5)}{=} \sum_i b_{y_i}\, dy_i\, \hat{e}_i \qquad\qquad dV \overset{(44.6)}{=} b_{y_1} b_{y_2} b_{y_3}\, dy_1\, dy_2\, dy_3$$

$$\Rightarrow \quad = \quad \hat{e}_r + \quad \hat{e}_\theta + \quad \hat{e}_\varphi \qquad\qquad = \quad dr\,d\theta\,d\varphi$$

Beispiel: F46

Volumen einer Kugel mit Radius R:

$$V = \int dV = \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi\; r^2\sin\theta$$

$$= \frac{1}{3}R^3 \underbrace{\left(-\cos\theta\right)\Big|_0^\pi}_{-(-1-1)=2} 2\pi = \frac{4\pi}{3}R^3$$

Bsp: $\vec{v}(\vec{r}) = \sum_i x_i\hat{e}_i = \hat{e}_r\, r$ sei Vektorfeld im 3 dimensionalen:

Überprüfen Sie Gauss'schen Satz (27.4) für die Kugelfläche $\vec{r} = R$, $\theta\in[0,\pi]$, $\varphi\in[0,2\pi]$

$$d\vec{A} = \hat{e}_r\; r\,d\theta\; r\sin\theta\,d\varphi$$

Skizze Seite 43

$$\int_{\text{Kugelfläche}} d\vec{A}\cdot\vec{v} = \underbrace{\int_0^\pi R\,d\theta \int_0^{2\pi} R\sin\theta\,d\varphi\;\hat{e}_r}_{d\vec{A}} \cdot \underbrace{\hat{e}_r R}_{\vec{v}} = R^3\cdot 2\cdot 2\pi$$

($\hat{e}_r\cdot\hat{e}_r = 1$)

$$\vec{\nabla}\cdot\vec{v} = \partial_1 x_1 + \partial_2 x_2 + \partial_3 x_3 = 3$$

$$\int_{\text{Kugelvolumen}} dV\,\vec{\nabla}\cdot\vec{v} = \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi\; r^2\sin\theta\cdot 3 = 4\pi R^3$$

= Gauss (27.4) ✓

Gradient in Krummlinigen Koordinatensystemen

$\varphi = \varphi(x_i) = \varphi(x_i(y_j))$

F47

Def. v. Gradient (in Cartesischen KS): $\vec{\nabla}\varphi \equiv \sum_i \hat{e}_i \frac{\partial \varphi}{\partial x_i}$ (1)

(44.3) $\frac{\partial \vec{r}}{\partial y_j} \frac{1}{b_{y_j}}$ (2)

Komponente v. $\vec{\nabla}\varphi$ in $\hat{e}_{y_k}$-Richtung: $(\vec{\nabla}\varphi)_{y_j} \equiv$

(3)

$=$

Kettenregel

Gradient in y_j-Koordinaten ausgedrückt:

$$\boxed{\vec{\nabla}\varphi = \sum_j \hat{e}_{y_j} \frac{1}{b_{y_j}} \frac{\partial}{\partial y_j} \varphi} \quad (4)$$

(5)