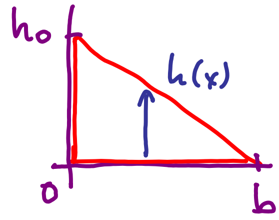
07-22.11.05 F32a

Flächenintegral: Integrationsgrenzen definieren die Fläche:

Beispiele: Fläche v. Dreieck:

$$h(x) = h_0(1 - x/b)$$



$$A = \int dA = \int_0^b dx \int_0^{h(x)} dy = \int_0^b dx\; h(x) = \int_0^b dx\; h_0(1 - x/b) =$$

$$= h_0\left(b - \tfrac{1}{2}\, b^2/b\right) = \tfrac{1}{2} h_0 b \qquad (1)$$

Dichte (pro Fläche) des Dreiecks sei gegeben durch: $\rho(x,y) = x^2 y$.

Was ist Masse des Dreiecks?

$$M = \int dA\, \rho(x,y) = \int_0^b dx \int_0^{h(1-x/b)} dy\; x^2 y = \int_0^b dx\; x^2\; \tfrac{1}{2} y^2 \Big|_0^{h(1-x/b)} \qquad (2)$$

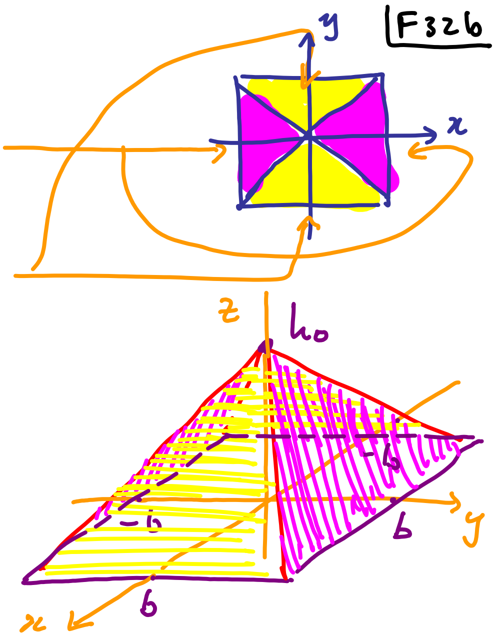
$$= \int_0^b dx\; x^2\; \frac{h_0^2}{2}(1 - x/b)^2 = \frac{h_0^2}{2} \int_0^b dx\; x^2 \left[1 - \frac{2x}{b} + \frac{x^2}{b^2}\right] \qquad (3)$$

$$= \frac{h_0^2}{2}\left[\frac{1}{3} b^3 - \frac{2}{b}\,\frac{1}{4} b^4 + \frac{1}{b^2}\,\frac{1}{5} b^5\right] = h_0^2 b^3\, \underbrace{\frac{1}{2}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right]}_{1/60} \qquad (4)$$

F32b

Volumenintegral: Volumen einer Pyramide

Höhe: $$h(x,y) = \begin{cases} h_0\left(1 - \frac{|x|}{b}\right) & \forall\ |x| > |y| \\ h_0\left(1 - \frac{|y|}{b}\right) & \forall\ |x| < |y| \end{cases}$$



$$\text{Volumen} = \int dV = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_0^{h(x,y)} dz$$

Integriere getrennt über die Bereiche $|x| > |y|$ und $|x| < |y|$:

$$\text{Volumen} = \int_{-b}^{b} dx \int_{-|x|}^{|x|} dy \int_0^{h_0(1-|x|/b)} dz + \int_{-b}^{b} dy \int_{-|y|}^{|y|} dx \int_0^{h_0(1-\frac{|y|}{b})} dz$$

liefert dasselbe

$$= \int_{-b}^{b} dx\; 2|x|\; h_0(1 - |x|/b) + \int_{-b}^{b} dy\; 2|y|\; h_0\left(1 - \frac{|y|}{b}\right)$$

$$= 2 \cdot 4 h_0 \int_0^b dx \left(x - \frac{x^2}{b}\right) = 8 h_0 \left[\tfrac{1}{2} b^2 - \tfrac{1}{3}\, b^3/b\right] = \frac{4}{3} h_0 b^2$$

Dichte pro Volumen der Pyramide sei $\rho(x,y,z) = z$

Masse: $M = \int dV \, \rho(x,y,z) = 4 \int_0^b dx \int_0^b dy \int_0^{h(x,y)} dz \; z$

$$= \int_{-b}^{b} dx \int_{-|x|}^{|x|} dy \int_0^{h_0(1-|x|/b)} dz \; z \quad + \quad \int_{-b}^{b} dy \int_{-|y|}^{|y|} dx \int_0^{h_0(1-\frac{|y|}{b})} dz \; z$$

liefert dasselbe

$$= \int_{-b}^{b} dx \; 2|x| \; \frac{1}{2} h_0^2 \left(1 - |x|/b\right)^2 \quad + \quad \int_{-b}^{b} dy \; 2|y| \; \frac{1}{2} h_0 \left(1 - |y|/b\right)^2$$

$$= 2 \; 2 \cdot h_0^2 \int_0^b dx \left(x - 2x^2/b + x^3/b^2 \right)$$

$$= 2 h_0^2 \left[\frac{1}{2} x - \frac{2}{3} x^3/b + \frac{1}{4} \frac{x^4}{b^2} \right]_0^b$$

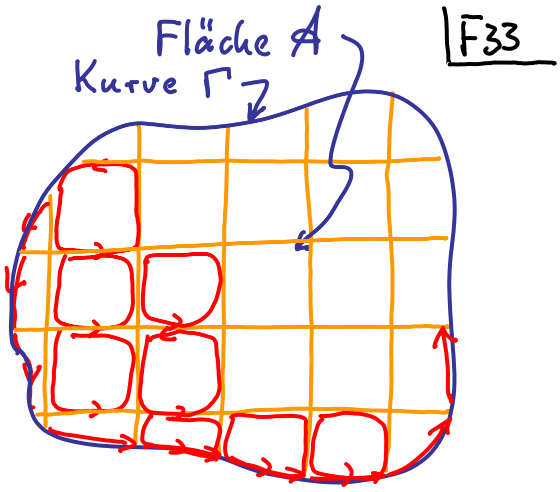
$$= 2 h_0^2 b \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] \qquad \frac{6-4+3}{12} = \frac{5}{12}$$

$$= \frac{5}{6} h_0^2 b$$

Stoke'sche Satz

07 – 21.11.05

Betrachte endliche Fläche A, umschlossen durch Kurve Γ:



Zirkulation um A: $\quad Z_A \stackrel{(30.1)}{\equiv} \oint d\vec{r} \cdot \vec{v} \qquad (1)$

alternativ

(Zirkulation pro Flächenelement) (Flächen-element)

$$\sum (\vec{\nabla} \times \vec{v}) \cdot \Delta \vec{A} \qquad (2)$$

über alle Flächenelemente

Zirkulation um ein typisches Flächenelement

$\lim \Delta A \to 0$

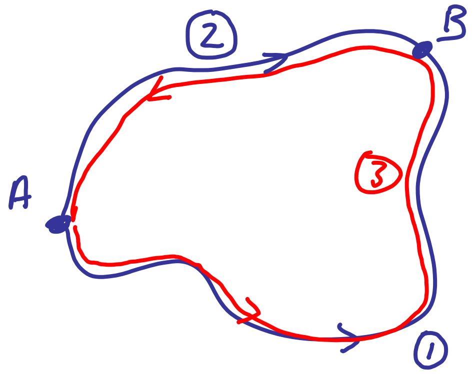
Flächenintegral   Linienintegral

$$\int_A d\vec{A} \cdot (\vec{\nabla} \times \vec{v}) \stackrel{(1)}{=} \oint d\vec{r} \cdot \vec{v} \qquad (3)$$

"Stoke'sche Satz"

gilt für "beliebige" Vektorfelder, beliebige A, Γ

Noch ein Beweis für Wegunabhängigkeit von $\int d\vec{r}\cdot\vec{\nabla}\varphi$



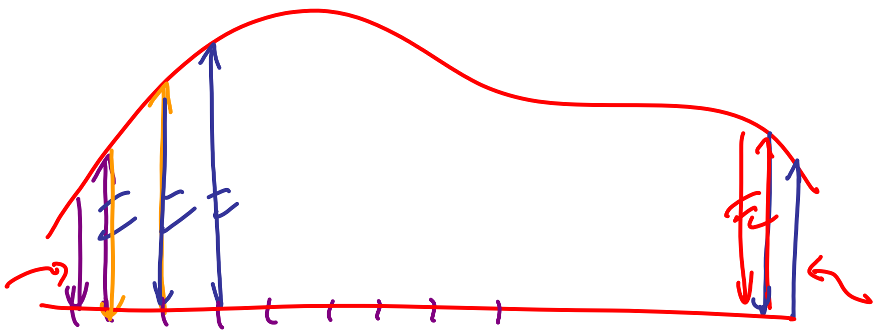
$$\underset{①: A\to B}{\int d\vec{r}\cdot\vec{\nabla}\varphi} - \underbrace{\underset{②: A\to B}{\int d\vec{r}\cdot\vec{\nabla}\varphi}}_{+\underset{②: B\to A}{\int d\vec{r}\cdot\vec{\nabla}\varphi}}$$

$$= \underset{③: A\to A}{\oint d\vec{r}\cdot\vec{\nabla}\varphi}$$

Stokes: $\overset{(33.3)}{=} \int d\vec{A}\cdot\left(\vec{\nabla}\times(\vec{\nabla}\varphi)\right)$

Gradientenfelder sind rotationsfrei: $\overset{(32.1)}{=} 0$

$\Longrightarrow$ Weg 1 und Weg 2 liefern ✓

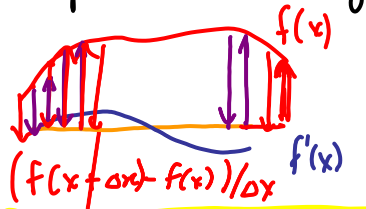


$$\int dx\, f'(x) = f(b) - f(a)$$

$$= \sum \Delta x \left[f(x+\Delta x) - f(x)\right].$$

Zusammenfassung: 3 Integralsätze der Vektoranalysis

Hauptsatz der Analysis:

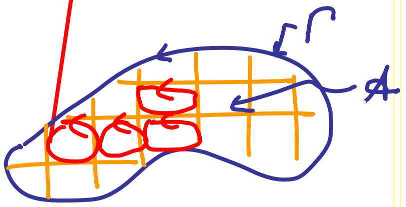


1-dim. Integral — 0-dim. Integral

$$\int_a^b dx\,(\partial_x f) = \lim_{\Delta x \to 0} \left(\frac{\Delta f}{\Delta x}\right) \Delta x = f(b) - f(a) \qquad (1)$$

Steigung = Änderung pro Abstand — Gesamtänderung

Stokes'sche Satz:

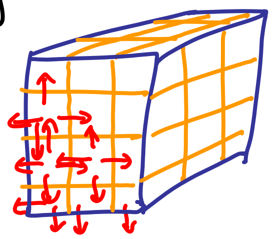


2-dim. Integral — 1-dim. Integral

$$\int_A d\vec{A} \cdot (\vec{\nabla} \times \vec{v}) = \lim_{\Delta A \to 0} \sum \Delta A \left(\frac{\Delta Z}{\Delta A}\right) = \int_\Gamma d\vec{r} \cdot \vec{v} \qquad (2)$$

Zirkulation pro Flächenelement — Gesamtzirkulation

Gauss'sche Satz:



3-dim. Integral — 2-dim. Integral

$$\int_V dV\,(\vec{\nabla} \cdot \vec{v}) = \lim_{\Delta V \to 0} \sum \Delta V \left(\frac{\Delta \text{Fluss}}{\Delta V}\right) = \int_A d\vec{A} \cdot \vec{v} \qquad (3)$$

Ausfluss pro Volumenelement — Gesamtausfluss

Krummlinige Koordinatensysteme : Ebene Polarkoordinaten in $\mathbb{E}^2$

Transformationsregeln:

$$x = \rho \cos\varphi \qquad (1a)$$

$$y = \rho \sin\varphi \qquad (1b)$$

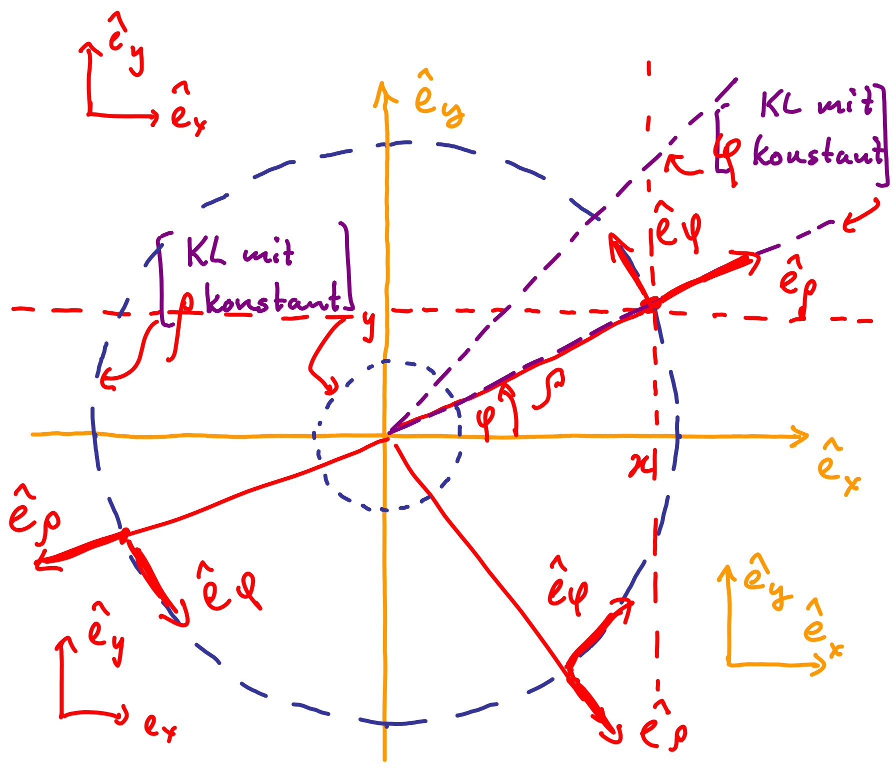
Umgekehrte Transformation:

$$\rho = \sqrt{x^2 + y^2}\,, \quad \rho \geq 0 \qquad (2a)$$

$$\varphi = \arctan(y/x) \in [0, 2\pi) \qquad (2b)$$

$\vec{r} = \vec{r}(x,y)$ mit $\left\{ \begin{matrix} x = \text{konst.} \\ y = \text{konst.} \end{matrix} \right\}$ definiert "Koordinatenlinie" (KL) parametrisiert durch $\left\{ \begin{matrix} y \\ x \end{matrix} \right\}$

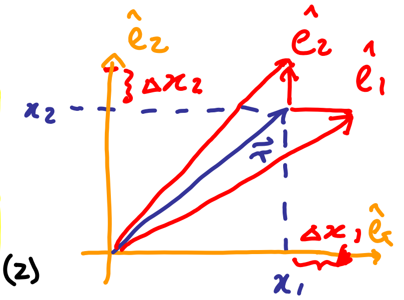
$\vec{r} = \vec{r}(\rho,\varphi)$ mit $\left\{ \begin{matrix} \rho = \text{konst.} \\ \varphi = \text{konst.} \end{matrix} \right\}$ definiert "Koordinatenlinie" (KL) parametrisiert durch $\left\{ \begin{matrix} \varphi \\ \rho \end{matrix} \right\}$



Lokales Zweibein: $\hat{e}_1 \equiv \begin{pmatrix}1\\0\\0\end{pmatrix}, \hat{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \hat{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}}$

Ortsvektor in cartesischen Koord: $\vec{r}(x, y) = x\,\hat{e}_x + y\,\hat{e}_y$ (1)

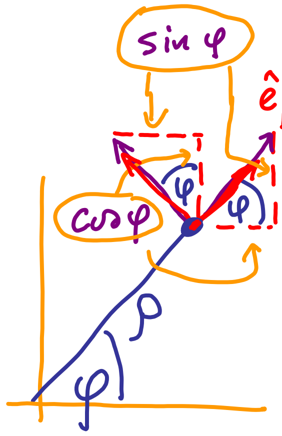
Lokales Zweibein, cartesisch: $\vec{r} \overset{(1)}{=} \sum_{i=1}^{2} x_i \hat{e}_i$, $\rightarrow$ $\boxed{\frac{\partial \vec{r}(x_1, x_2)}{\partial x_i} = \hat{e}_i = \text{parallel zur KL}}$ (2)



$\Rightarrow$ **Basisvektoren sind tangential zu KL**

Dieselbe Konstruktion in Polarkoordinaten:

Ortsvektor [= (1)] in Polarkoordinaten: $\vec{r}(\rho, \varphi) \overset{(37.1)}{=} \underbrace{\rho\cos\varphi}_{x_1}\,\hat{e}_x + \underbrace{\rho\sin\varphi}_{x_2}\,\hat{e}_y$ (3)



Lokales Zweibein, in Polarkoordinaten: [per Konstruktion tangential zu KL]

$\frac{\partial \vec{r}(\rho,\varphi)}{\partial \rho} \overset{(3)}{=} \underbrace{\cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y}_{\text{Betrag } = 1} \equiv \hat{e}_\rho$ (4)

$\frac{\partial \vec{r}(\rho,\varphi)}{\partial \varphi} \overset{(3)}{=} \underbrace{-\rho\sin\varphi\,\hat{e}_x + \rho\cos\varphi\,\hat{e}_y}_{\text{Betrag } = \rho} \equiv \rho\,\hat{e}_\varphi$ (5)

Allgemein: $\hat{e}_\rho, \hat{e}_\varphi$ bilden ein lokales (ortsabhängiges) orthonormales Zweibein:

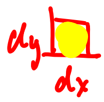
$$\hat{e}_\rho \equiv \frac{\partial \vec{r}}{\partial \rho} \Big/ \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \cos\varphi\,\hat{e}_x + \sin\,\hat{e}_y \quad (1)$$

$$\hat{e}_\varphi \equiv \frac{\partial \vec{r}}{\partial \varphi} \Big/ \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = -\sin\varphi\,\hat{e}_x + \cos\,\hat{e}_y \quad (2)$$

$$\vec{r} = x\hat{e}_x + y\hat{e}_y = \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y = \rho\,\hat{e}_\rho \quad (3)$$

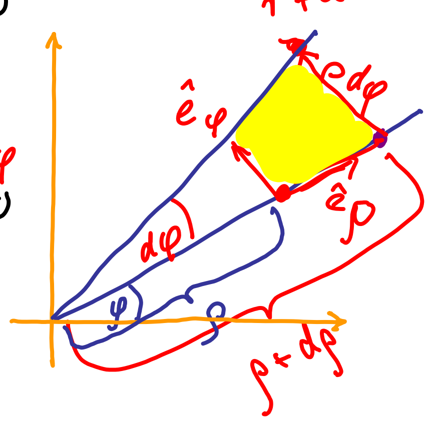
WICHTIG!!

Wegelement: Skizze: siehe Seite 37.



cartesisch: $d\vec{r} = \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy = \hat{e}_x\,dx + \hat{e}_y\,dy$ (4)

polar: $= \frac{\partial \vec{r}}{\partial \rho}d\rho + \frac{\partial \vec{r}}{\partial \varphi}d\varphi = \underbrace{\left|\frac{\partial \vec{r}}{\partial \rho}\right|}_{=1}\hat{e}_\rho\,d\rho + \underbrace{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|}_{\rho}\hat{e}_\varphi\,d\varphi$ (5)

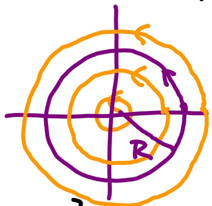


$\Rightarrow$ $\boxed{d\vec{r} = d\rho\,\hat{e}_\rho + \rho\,d\varphi\,\hat{e}_\varphi}$ (6)

Dimensionen: ($d\vec{r}$, $d\rho$, ρ:) Länge; ($d\varphi$:) dimensionslos!

Flächenelement: $dA = dx\,dy = \rho\,d\rho\,d\varphi$ (7)

Bsp: Fläche eines Kreises: $A = \int dA = \int_0^r \rho\, d\rho \int_0^{2\pi} d\varphi = \frac{1}{2} r^2 2\pi = \pi r^2$ (1)



Bsp: $\vec{v}(\vec{r}) = r\,\hat{e}_\varphi$ sei: Vektorfeld im 2-dimensionalen:

Überprüfen Sie Stoke'schen Satz (33.3) für den Kreisweg $\vec{r} = R,\ \varphi \in [0, 2\pi]$

$d\vec{r} = R\,d\varphi\,\hat{e}_\varphi$ (2)

$$\oint_{\text{Rand des Kreises}} d\vec{r}\cdot\vec{v} = \int_0^{2\pi} \underbrace{R\,d\varphi\,\hat{e}_\varphi}_{d\vec{r}} \cdot \underbrace{(R\,\hat{e}_\varphi)}_{\vec{v}(R)} = R^2 \int_0^{2\pi} d\varphi = 2\pi R^2$$

$\vec{\nabla}\times\vec{v} = ?$

$$\vec{v}(\vec{r}) \overset{(39.2)}{=} r\,(-\sin\varphi\,\hat{e}_x + \cos\,\hat{e}_y) = -y\,\hat{e}_x + x\,\hat{e}_y$$

$$\vec{\nabla}\times\vec{v} = (\partial_2 v_3 - \partial_3 v_2,\ \partial_3 v_1 - \partial_1 v_3,\ \partial_1 v_2 - \partial_2 v_1)$$

$$= (0, 0, 2) = 2\,\hat{e}_z$$

Stokes (33.3)

$(\rho\, d\rho\, d\varphi\, \hat{e}_z)$

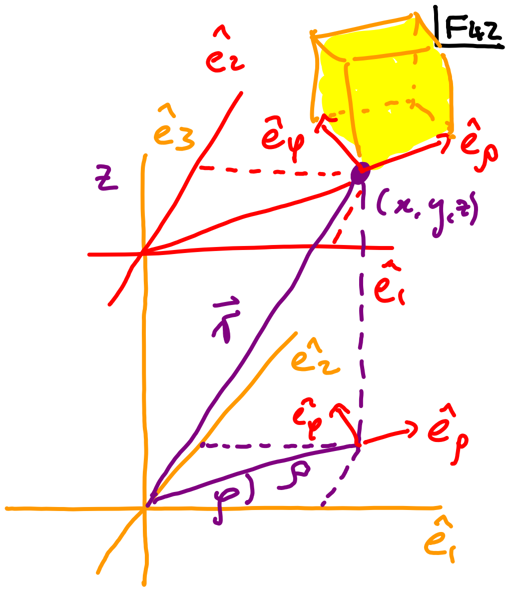
$$\int_{\text{Fläche des Kreises}} d\vec{A}\cdot(\vec{\nabla}\times\vec{v}) = \int_0^R \rho\, d\rho \int_0^{2\pi} d\varphi\ \underbrace{\hat{e}_z\cdot(\vec{\nabla}\times\vec{v})}_{2} = \left(\tfrac{1}{2}R^2\right) 2\pi\cdot 2 = 2\pi R^2$$

Polarkoordinaten in 3 Dimension

Def: $x = \rho\cos\varphi$, (1a) $\rho = \sqrt{x^2+y^2}$ (1d)

$y = \rho\sin\varphi$ (1b) $\varphi = \arctan(y/x) \in [0, 2\pi)$ (1e)

$z = z$ (1c) $z = z$ (1f)



Ortsvektor: $\vec{r} = \rho\cos\varphi\,\hat{e}_x + \rho\sin\varphi\,\hat{e}_y + z\,\hat{e}_z$ (2)

Polares Dreibein:
(analog zu 2-D)

$$\hat{e}_\rho \overset{(2)}{\equiv} \frac{\partial\vec{r}}{\partial\rho}\Big/\left|\frac{\partial\vec{r}}{\partial\rho}\right| = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y \quad (3a)$$

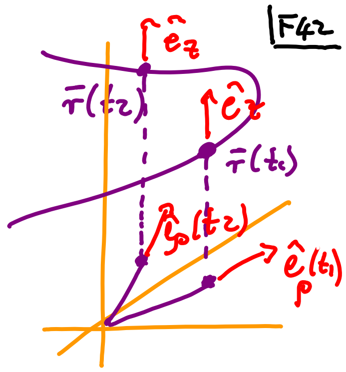
$$\hat{e}_\varphi \overset{(2)}{\equiv} \frac{\partial\vec{r}}{\partial\varphi}\Big/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = -\sin\varphi\,\hat{e}_x + \cos\,\hat{e}_y \quad (3b)$$

$$\hat{e}_z \overset{(2)}{\equiv} \frac{\partial\vec{r}}{\partial z}\Big/\left|\frac{\partial\vec{r}}{\partial\varphi}\right| = \hat{e}_z \quad (3c)$$

Wegelement: $d\vec{r} = d\rho\,\hat{e}_\rho + \rho\, d\varphi\,\hat{e}_\varphi + dz\,\hat{e}_z$ (4)

Volumenelement: $dV = \rho\, d\rho\, d\varphi\, dz$ (5)

Anwendung auf Bahnkurve:



zeitunabhängig

Ortsvektor: $\vec{r}(t) = x(t)\,\hat{e}_x + y(t)\,\hat{e}_y + z(t)\,\hat{e}_z$ (1)

zeitabhängig! $= \rho(t)\,\hat{e}_\rho(t) + z(t)\,\hat{e}_z$ (2)

Produktregel

Geschwindigkeitsvektor: $\vec{v}(t) = \frac{d\vec{r}}{dt} \overset{(2)}{=} \dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\hat{e}}_\rho + \dot{z}\,\hat{e}_z$ (3)

Berechne $\dot{\hat{e}}_\rho$, $\dot{\hat{e}}_\varphi$ in Cartesischen Koord., wo Basisvekt. zeitunabhängig sind:

$$\dot{\hat{e}}_\rho \overset{(39.1)}{=} \frac{d}{dt}\left[\cos\varphi_{(t)}\,\hat{e}_x + \sin\varphi_{(t)}\,\hat{e}_y\right] = \dot{\varphi}\underbrace{\left[-\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y\right]}_{\hat{e}_\varphi}$$

$$\dot{\hat{e}}_\varphi \overset{(39.2)}{=} \frac{d}{dt}\left[-\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y\right] = -\dot{\varphi}\underbrace{\left[\cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y\right]}_{\hat{e}_\rho}$$

$$\Rightarrow \quad \vec{v}(t) = \dot{\rho}\,\hat{e}_\rho + \rho\dot{\varphi}\,\hat{e}_\varphi + \dot{z}\,\hat{e}_z$$

Beschleunigung: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot{\rho}\,\hat{e}_\rho + \dot{\rho}\,\underbrace{\dot{\hat{e}}_\rho}_{\dot{\varphi}\hat{e}_\varphi} + \dot{\rho}\dot{\varphi}\,\hat{e}_\varphi + \rho\ddot{\varphi}\,\hat{e}_\varphi + \rho\dot{\varphi}\,\underbrace{\dot{\hat{e}}_\varphi}_{-\dot{\varphi}\hat{e}_\rho} + \ddot{z}\,\hat{e}_z$

$$= (\ddot{\rho} - \rho\dot{\varphi}^2)\,\hat{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\,\hat{e}_\varphi + \ddot{z}\,\hat{e}_z$$

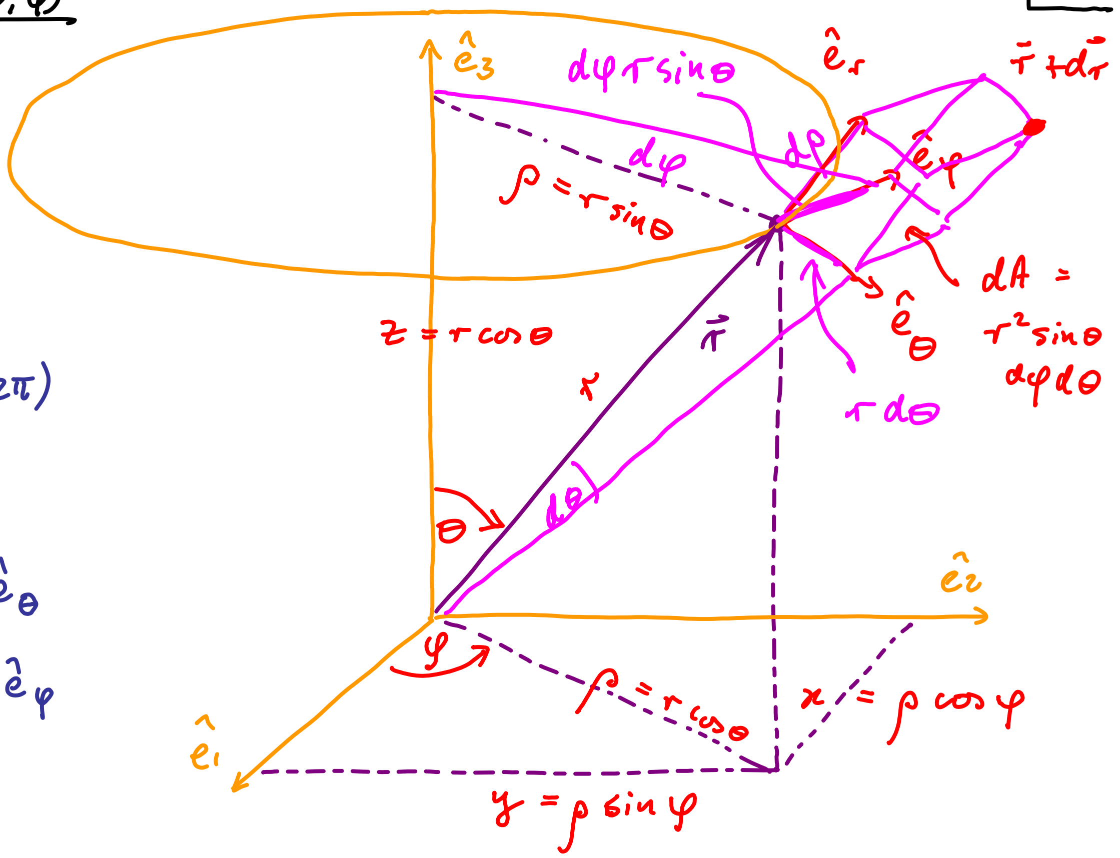
Kugelkoordinaten: (r, θ, φ)

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$



Wegelement:

$$d\vec{r} = dr\,\hat{e}_r + d\theta\, r\,\hat{e}_\theta + d\varphi\, r\sin\theta\,\hat{e}_\varphi$$

Volumenelement:

$$dV = r^2\sin\theta\, dr\, d\theta\, d\varphi$$

Allgemeine Koordinatentransformationen in 3D

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Def. der Transf: $x_i = x_i(y_j)$, $i = 1,2,3$; $j = 1,2,3$ (1)

↑ Cartesisch ↑ krummlinig

Ortsvektor: $\vec{r} = \sum_i x_i \hat{e}_i = \sum_i x_i(y_j)\, \hat{e}_i$ (2)

Normierungsfaktor

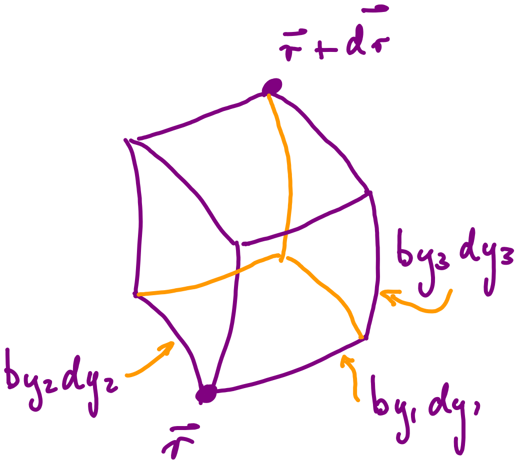
Lokales Dreibein: $\hat{e}_{y_i} \equiv \frac{\partial \vec{r}}{\partial y_i} \frac{1}{b_{y_i}}$, mit $b_{y_i} = \left| \frac{\partial \vec{r}}{\partial y_i} \right|$ $i = 1,2,3$ (3)

Wegelement: $d\vec{r} = \sum_i \frac{\partial \vec{r}}{\partial y_i} dy_i$ (4)

$d\vec{r} = \sum_i dx_i \hat{e}_i$ $\overset{(3)}{=} \sum_i b_{y_i} dy_i \, \hat{e}_{y_i}$ (5)

Dimension: Länge

b_{y_i}



Volumenelement: $dV = dx_1 dx_2 dx_3$

$= b_{y_1} b_{y_2} b_{y_3} \, dy_1 dy_2 dy_3$ (6)

Beispiel Kugelkoordinaten:

$y_1 \equiv r, \quad y_2 = \theta, \quad y_3 = \varphi$

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$x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$ (1)

$\vec{r} = \sum_i x_i \hat{e}_i = r \left[\sin\theta \cos\varphi \, \hat{e}_1 + \sin\theta \sin\varphi \, \hat{e}_2 + \cos\theta \, \hat{e}_3 \right]$ (2)

$\frac{\partial \vec{r}}{\partial r} \overset{(2)}{=} \left[\sin\theta \cos\varphi \, \hat{e}_1 + \sin\theta \sin\varphi \, \hat{e}_2 + \cos\theta \, \hat{e}_3 \right] \equiv b_r \, \hat{e}_r$ (3)

$\frac{\partial \vec{r}}{\partial \theta} \overset{(2)}{=} r \left[\cos\theta \cos\varphi \, \hat{e}_1 + \cos\theta \sin\varphi \, \hat{e}_2 - \sin\theta \, \hat{e}_3 \right] \equiv b_\theta \, \hat{e}_\theta$ (4)

$\frac{\partial \vec{r}}{\partial \varphi} \overset{(2)}{=} r \left[-\sin\theta \sin\varphi \, \hat{e}_1 + \sin\theta \cos\varphi \, \hat{e}_2 \right] \equiv b_\varphi \, \hat{e}_\varphi$ (5)

mit $b_r = 1$, $b_\theta = r$ $b_\varphi = r \sin\theta$ (6)

$d\vec{r} \overset{(44.5)}{=} \sum_i b_{y_i} dy_i \, \hat{e}_i$

$\Rightarrow \quad = dr \, \hat{e}_r + r \, d\theta \, \hat{e}_\theta + r \sin\theta \, d\varphi \, \hat{e}_\varphi$

$dV \overset{(44.6)}{=} b_{y_1} b_{y_2} b_{y_3} \, dy_1 dy_2 dy_3$

$= r^2 \sin\theta \, dr \, d\theta \, d\varphi$

Beispiel:

Volumen einer Kugel mit Radius R:

$$V = \int dV = \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; r^2 \sin\theta$$

$$= \frac{1}{3} R^3 \underbrace{\left(-\cos\theta\right)_0^\pi}_{-(-1-1)=2} 2\pi = \frac{4\pi}{3} R^3$$

Bsp: $\vec{v}(\vec{r}) = \sum_i x_i \hat{e}_i = \hat{e}_r r$ sei Vektorfeld im 3 dimensionalen:

Überprüfen Sie Gauss'schen Satz (27.4) für die Kugelfläche $\vec{r} = R$, $\theta \in [0,\pi]$, $\varphi \in [0, 2\pi]$

$$d\vec{A} = \hat{e}_r \; r\,d\theta \; r\sin\theta\,d\varphi$$

$$\int_{\text{Kugelfläche}} d\vec{A}\cdot\vec{v} = \underbrace{\int_0^\pi R\,d\theta \int_0^{2\pi} R\sin\theta\,d\varphi \; \hat{e}_r}_{d\vec{A}} \cdot \underbrace{\hat{e}_r R}_{\vec{v}} = R^3 \cdot 2 \cdot 2\pi$$

($\hat{e}_r \cdot \hat{e}_r = 1$)

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$$\vec{\nabla}\cdot\vec{v} = \partial_1 x_1 + \partial_2 x_2 + \partial_3 x_3 = 3$$

$$\int_{\text{Kugelvolumen}} dV\, \vec{\nabla}\cdot\vec{v} = \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; r^2 \sin\theta \cdot 3 = 4\pi R^3$$

= Gauss (27.4)