

Def. der Transf.: $x_i = x_i(y_j)$, $i = 1, 2, 3$; $j = 1, 2, 3$ (1)

↑
Cartesisch

↑
krümmung

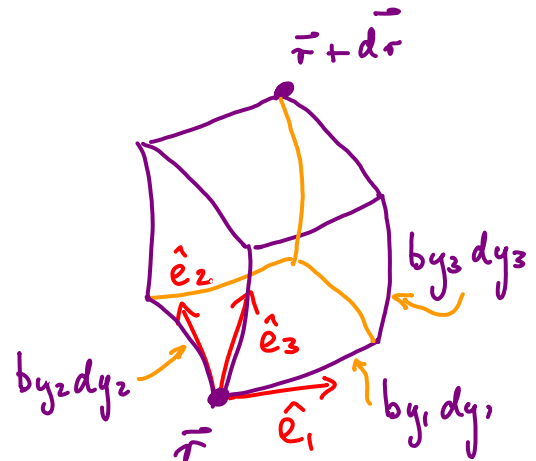
Ortsvektor: $\vec{r} = \sum_i x_i \hat{e}_i$ (2)

Lokales Dreibein: $\hat{e}_{y_i} \equiv \frac{\partial \vec{r}}{\partial y_i} \frac{1}{b_{y_i}}$, mit $b_{y_i} \equiv \left| \frac{\partial \vec{r}}{\partial y_i} \right|$ $i = 1, 2, 3$ (3)

↑
Normierungsfaktor

Wegelement: $d\vec{r} = \sum_i \frac{\partial \vec{r}}{\partial y_i} dy_i$ (4)

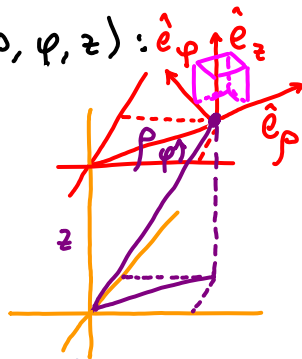
$\stackrel{(3)}{=} \sum_i \underbrace{b_{y_i} dy_i}_{\text{Dimension: Länge}} \hat{e}_i$ (5)



Volumenelement: $dV = dx_1 dx_2 dx_3$
 $= b_{y_1} b_{y_2} b_{y_3} dy_1 dy_2 dy_3$ (6)

3D Polarkoordinaten:

$$\begin{aligned} x &= \rho \cos \varphi, \\ y &= \rho \sin \varphi, \\ z &= z \end{aligned}$$



$$\vec{r} = \rho \cos \varphi \hat{e}_x + \rho \sin \varphi \hat{e}_y + z \hat{e}_z$$

$$\hat{e}_\rho = \cos \varphi \hat{e}_x + \sin \varphi \hat{e}_y$$

$$\hat{e}_\varphi = -\sin \varphi \hat{e}_x + \cos \varphi \hat{e}_y$$

$$\hat{e}_z = \hat{e}_z$$

$$b_\rho = 1, \quad b_\varphi = \rho, \quad b_z = 1$$

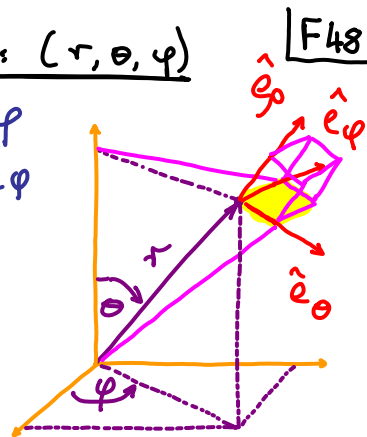
$$d\vec{r} = d\rho \hat{e}_\rho + \rho d\varphi \hat{e}_\varphi + dz \hat{e}_z$$

$$dV = \rho d\rho d\varphi dz$$

Kugelkoordinaten: (r, theta, phi)

$$\begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta \end{aligned}$$

$$\vec{r} = r \hat{e}_r$$



$$\hat{e}_r = \sin \theta \cos \varphi \hat{e}_1 + \sin \theta \sin \varphi \hat{e}_2 + \cos \theta \hat{e}_3$$

$$\hat{e}_\theta = \cos \theta \cos \varphi \hat{e}_1 + \cos \theta \sin \varphi \hat{e}_2 - \sin \theta \hat{e}_3$$

$$\hat{e}_\varphi = -\sin \theta \sin \varphi \hat{e}_1 + \sin \theta \cos \varphi \hat{e}_2$$

$$b_r = 1, \quad b_\theta = r, \quad b_\varphi = r \sin \theta$$

$$d\vec{r} = dr \hat{e}_r + r d\theta \hat{e}_\theta + r \sin \theta d\varphi \hat{e}_\varphi$$

$$dV = r^2 \sin \theta dr d\theta d\varphi$$

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\ \hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \\ \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \\ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \end{cases}$$

Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \quad \begin{cases} \hat{x} = \cos \phi \hat{s} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \phi \hat{s} + \cos \phi \hat{\phi} \\ \hat{z} = \hat{z} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \quad \begin{cases} \hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y} \\ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \\ \hat{z} = \hat{z} \end{cases}$$

VECTOR DERIVATIVES

$t(x, y, z) =$: Skalares Feld
 $\vec{v}(x, y, z) =$: Vektorfeld

Cartesian. $d\mathbf{l} = dx \hat{x} + dy \hat{y} + dz \hat{z}$; $d\tau = dx dy dz$

Gradient: $\nabla t = \frac{\partial t}{\partial x} \hat{x} + \frac{\partial t}{\partial y} \hat{y} + \frac{\partial t}{\partial z} \hat{z}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}$

Laplacian: $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$; $d\tau = r^2 \sin \theta dr d\theta d\phi$

Gradient: $\nabla t = \frac{\partial t}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\phi}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$

Curl: $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi}$

Laplacian: $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$; $d\tau = s ds d\phi dz$

Gradient: $\nabla t = \frac{\partial t}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\phi} + \frac{\partial t}{\partial z} \hat{z}$

Divergence: $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\phi} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_\phi}{\partial \phi} \right] \hat{z}$

Laplacian: $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

Gradient in Krummlinigen Koordinatensystemen

$$S = S(x_i) = S(x_i(y_j)) \quad |F47$$

Def. v. Gradient
(in Cartesischen KS):

$$\vec{\nabla} S \equiv \sum_i \hat{e}_i \frac{\partial S}{\partial x_i} \quad (1)$$

$$(44.3) \quad \frac{\partial \tau}{\partial y_j} \frac{1}{b_{y_j}} \rightarrow \quad (2)$$

Komponente v. $\vec{\nabla} \varphi$
in $\hat{e}_{y_k}$ -Richtung:

$$(\vec{\nabla} S)_{y_j} = \quad (3)$$

$$=$$

Kettenregel

Gradient in y_j -
Koordinaten
ausgedrückt:

$$\vec{\nabla} S = \sum_j \hat{e}_{y_j} \frac{1}{b_{y_j}} \frac{\partial}{\partial y_j} S \quad (4)$$

$$\vec{\nabla} = \sum_j \hat{e}_{y_j} \frac{1}{b_{y_j}} \frac{\partial}{\partial y_j} \quad (5)$$

Zylinderkoordinaten:
 $b_\rho = 1, b_\varphi = \rho, b_z = 1$

$$\vec{\nabla} = \quad (47.4)$$

Kugelkoordinaten:
 $b_r = 1, b_\theta = r, b_\varphi = r \sin \theta$

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \quad (2)$$

Divergenz in krummlinigen Koordinaten

in Cartesischen Koordinaten
" Krummlinigen "

Betrachte Vektorfeld:

$$\vec{v} = v_i \hat{e}_i = v_{y_j} \hat{e}_{y_j} \quad (3)$$

$$\text{Divergenz cartesisch: } \vec{\nabla} \cdot \vec{v} = (\hat{e}_i \partial_i) \cdot (v_j \hat{e}_j) = \quad (4)$$

krummlinig:

$$= \sum_{ij} \left(\frac{\hat{e}_{y_i}}{b_{y_i}} \partial_{y_i} \right) \cdot \left(v_{y_j} \hat{e}_{y_j} \right) \quad (5)$$

Ortsabhängig!! $\Rightarrow$
Abl. wirkt auf
Komponenten
und auf Dreibein!!

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{b_{y_1} b_{y_2} b_{y_3}} \left[\partial_{y_1} (v_{y_1} b_{y_2} b_{y_3}) + \partial_{y_2} (b_{y_1} v_{y_2} b_{y_3}) + \partial_{y_3} (b_{y_1} b_{y_2} v_{y_3}) \right] \quad (6)$$

Stunden später...
(Siehe Notiz)

Wegelement:

$$\Delta \vec{r} = \sum_i b_{y_i} \Delta y_i \hat{e}_{y_i}$$

Flächenelemente:

$$\Delta A_{1,2} = \pm (b_{y_2} \Delta y_2 b_{y_3} \Delta y_3) \hat{e}_{y_1}$$

$$\Delta A_{3,4} = \pm (b_{y_1} \Delta y_1 b_{y_3} \Delta y_3) \hat{e}_{y_2}$$

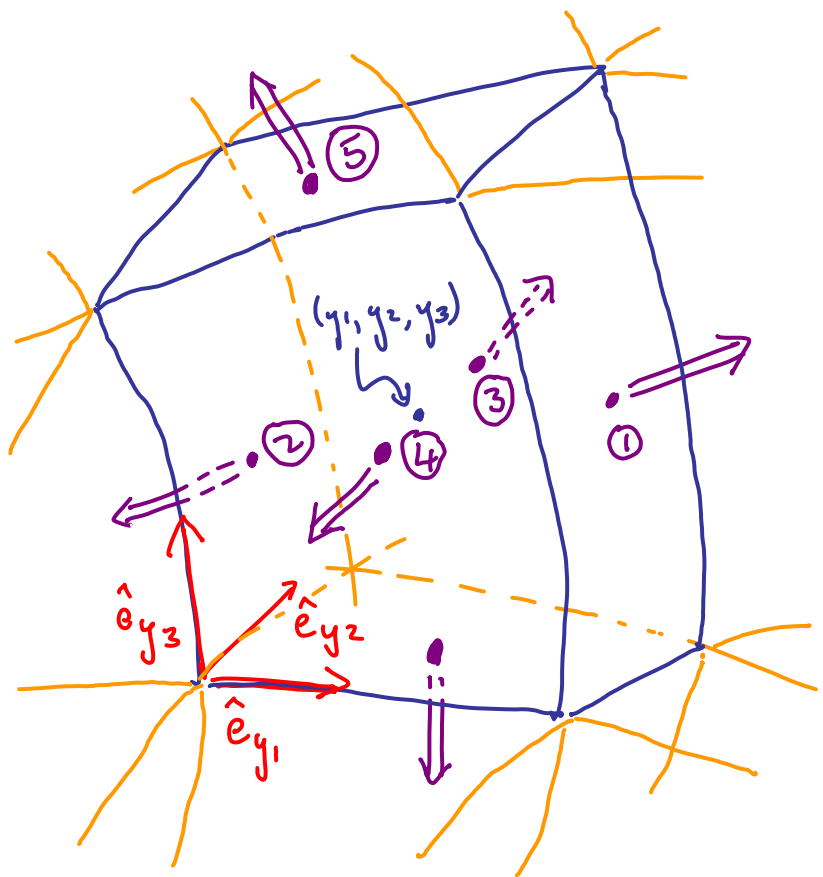
$$\Delta A_{5,6} = \pm (b_{y_1} \Delta y_1 b_{y_2} \Delta y_2) \hat{e}_{y_3}$$

keine b's hier!

$$\textcircled{1}, \textcircled{2} = (y_1 \pm \frac{\Delta y_1}{2}, y_2, y_3)$$

$$\textcircled{3}, \textcircled{4} = (y_1, y_2 \pm \frac{\Delta y_2}{2}, y_3)$$

$$\textcircled{5}, \textcircled{6} = (y_1, y_2, y_3 \pm \frac{\Delta y_3}{2})$$



$$\text{Divergenz:} = \left(\frac{\text{Ausfluss}}{\text{Volumen } \Delta V} \right) = \frac{\sum_{i=1}^6 (\vec{v} \cdot \Delta \vec{A})_{\textcircled{i}}}{(b_{y_1} \Delta y_1) (b_{y_2} \Delta y_2) (b_{y_3} \Delta y_3)} \quad \text{F50} \quad (1)$$

$$= \left[\frac{\quad}{(b_{y_1} \Delta y_1) (b_{y_2} \Delta y_2) (b_{y_3} \Delta y_3)} \right] \Delta y_2 \Delta y_3 \quad (2)$$

+ analog für $\textcircled{3} - \textcircled{4}$ und $\textcircled{5} - \textcircled{6}$

$$\vec{v} \cdot \vec{v} = \frac{1}{b_{y_1} b_{y_2} b_{y_3}} \left[\partial_{y_1} (v_{y_1} b_{y_2} b_{y_3}) + \partial_{y_2} (b_{y_1} v_{y_2} b_{y_3}) + \partial_{y_3} (b_{y_1} b_{y_2} v_{y_3}) \right] \quad (3)$$

Beispiele:

|FSI

Zylinderkoordinaten:

$$b_\rho = 1, b_\varphi = \rho, b_z = 1$$

$$\vec{\nabla} \cdot \vec{v} =$$

(1)

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{\rho} \partial_\rho (\rho v_\rho) + \frac{1}{\rho} \partial_\varphi v_\varphi + \partial_z v_z \quad (2)$$

Kugelkoordinaten:

$$b_r = 1, b_\theta = r, b_\varphi = r \sin \theta$$

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2 \sin \theta} \left[\partial_r (r^2 \sin \theta v_r) + \partial_\theta (r \sin \theta v_\theta) + \partial_\varphi (r v_\varphi) \right] \quad (3)$$

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \partial_r (r^2 v_r) + \frac{1}{r \sin \theta} \partial_\theta (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \partial_\varphi v_\varphi \quad (4)$$

Anwendung:

$$\text{Sei } \vec{v} = \vec{r} \Rightarrow \vec{\nabla} \cdot \vec{v} =$$

$$\text{Sei } \vec{v} = \hat{e}_\varphi \Rightarrow \vec{\nabla} \cdot \vec{v} =$$

Rotation: $\vec{\nabla} \times \vec{v} = (\hat{e}_i \partial_i) \times (v_j \hat{e}_j) = (\hat{e}_{y_i} \partial_{y_i}) \times (v_{y_j} \hat{e}_{y_j}) \quad |FS2$

(1)

Stunden später...

$$= \frac{1}{b_{y_2} b_{y_3}} \hat{e}_{y_1} \left[\partial_{y_2} (b_{y_3} v_{y_3}) - \partial_{y_3} (b_{y_2} v_{y_2}) \right]$$

$$+ \frac{1}{b_{y_3} b_{y_1}} \hat{e}_{y_2} \left[\partial_{y_3} (b_{y_1} v_{y_1}) - \partial_{y_1} (b_{y_3} v_{y_3}) \right]$$

(2)

$$+ \frac{1}{b_{y_1} b_{y_2}} \hat{e}_{y_3} \left[\partial_{y_1} (b_{y_2} v_{y_2}) - \partial_{y_2} (b_{y_1} v_{y_1}) \right]$$

Zylinderkoordinaten: $b_\rho = 1, b_\varphi = \rho, b_z = 1$

$$\vec{\nabla} \times \vec{v} = \frac{1}{\rho} \hat{e}_\rho [\partial_\varphi v_z - \partial_z (\rho v_\varphi)] + \hat{e}_\varphi [\partial_z v_\rho - \partial_\rho v_z] + \frac{1}{\rho} \hat{e}_z [\partial_\rho (\rho v_\varphi) - \partial_\varphi v_\rho]$$

Anwendung: Sei $\vec{v} = \hat{e}_\varphi$

$$\vec{\nabla} \times \vec{v} =$$

Kugelkoordinaten: Selber nachrechnen!

Geometrische Interpretation: (betrachte nur z-Komponente)

$$(\text{Rotation})_3 = \frac{\text{Zirkulation}}{\text{Fläche } \Delta A_3}$$

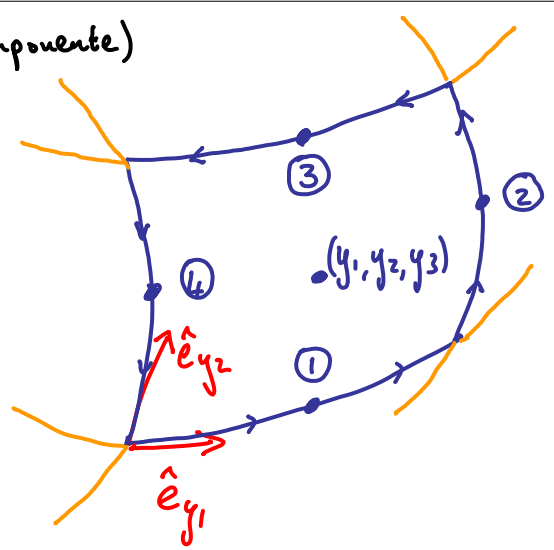
$$= \frac{1}{\Delta A} \sum_{i=1}^4 (\vec{v} \cdot \Delta \vec{r})_{(i)}$$

$$= \frac{1}{b_{y_1} b_{y_2} \Delta y_1 \Delta y_2} \left[\right] \Delta y_1$$

$$\left[(v_{y_2} b_{y_2})_{(2)} - (v_{y_2} b_{y_2})_{(4)} \right] \Delta y_2$$

$$(\vec{\nabla} \times \vec{v})_3 = \frac{1}{b_{y_1} b_{y_2}} \left[\partial_1 (b_{y_2} v_2) - \partial_2 (b_{y_1} v_1) \right]$$

analog für $(\vec{\nabla} \times \vec{v})_{1,2}$.



$$(1), (3) = (y_1, y_2 \pm \frac{\Delta y_2}{2})$$

$$(2), (4) = (y_1 \pm \frac{\Delta y_1}{2}, y_2)$$

Wegelemente:

$$\Delta \vec{r}_{1,3} = \pm b_{y_1} \Delta y_1 \hat{e}_{y_1}$$

$$\Delta \vec{r}_{2,4} = \pm b_{y_2} \Delta y_2 \hat{e}_{y_2}$$

$$\text{Flächenelement: } \Delta A_3 = b_{y_1} \Delta y_1 b_{y_2} \Delta y_2$$

Matrizen

[M]

Schöne Zusammenfassung: Nolting I

(hier ganz knapp; ausführlich $\Rightarrow$ lineare Algebra)

Definition einer

$m \times n$ Matrix

Anzahl Zeilen, Spalten

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & \vdots \\ \vdots & \vdots & A_{33} & \vdots \\ A_{m1} & \dots & \dots & A_{mn} \end{bmatrix} = (A_{ij})_{\substack{i=1, \dots, \\ j=1, \dots,}} \quad (1)$$

"Quadratische Matrix":

Falls

Matrix-Elemente

(2)

"Null-Matrix":

$$A = 0 \Leftrightarrow A_{ij} = 0 \quad \forall \quad i, j \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & & & & \\ \vdots & & & & \\ 0 & \dots & & & 0 \end{bmatrix}$$

"Diagonal-Matrix":
(für $m=n$)

$$d_{ij} = \begin{cases} d_i & i=j \\ 0 & i \neq j \end{cases} \Leftrightarrow \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{pmatrix} \quad \text{M2} \quad (1)$$

"Einheitsmatrix":
(für $m=n$)

$$E_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \Leftrightarrow \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \quad (2)$$

$A = (A_{ij})$ sei
gegebene $m \times n$ -Matrix;

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{m1} & \dots & & A_{mn} \end{pmatrix} \quad \text{z.B.} \quad A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad (3)$$

"Transponierte Matrix"

ist $A^T = (A_{ji}^T) = (A_{ji})$

vertausche Zeilen $\Leftrightarrow$ Spalten

$$A^T = \begin{pmatrix} A & A & \dots & A \\ A & & & \\ \vdots & & & \\ A & \dots & & A \end{pmatrix} \quad A^T = \quad (4)$$

Rechenregeln:

Seien $A = (A_{ij})$ und $B = (B_{ij})$ zwei $m \times n$ -Matrizen M3
(1)

Addition:

$$C = A + B := \quad \text{mit}$$

Skalare Multiplikation:

$$\lambda A = \quad (2)$$

Beispiele: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}, \quad 3 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \quad (3)$

Seien $A = (A_{ij})$:

und $B = (B_{ij})$:

(4)

Matrixmultiplikation:

$$C = A \cdot B = \quad , \text{ mit}$$

C_{ij} = "Skalarprodukt" aus i -tem "Zeilenvektor" und j -tem "Spaltenvektor"

Explizit:

m Zeilen

i-ter Zeilenvektor

j-ter Spaltenvektor

r Spalten

M4

$$\begin{pmatrix} - & - & - & - \\ - & - & - & - \\ A_{i1} & A_{i2} & \dots & A_{in} \\ - & - & - & - \\ - & - & - & - \end{pmatrix} \cdot \begin{pmatrix} | & | & B_{1j} & | & | \\ | & | & B_{2j} & | & | \\ | & | & \vdots & | & | \\ | & | & B_{nj} & | & | \end{pmatrix} = \begin{pmatrix} | & & & | \\ \vdots & & & \vdots \\ - & C_{ij} & - & - \\ \vdots & & & \vdots \end{pmatrix}$$

Zeilen, Spalten

Beispiel:

$$m=3, n=2, r=2$$

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 \cdot 0 + 5 \cdot 2 & 4 \cdot 1 + 5 \cdot 1 \\ 1 \cdot 0 + 0 \cdot 2 & 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 10 & 9 \\ 0 & 1 \end{pmatrix}$$

Allgemein gilt:

Matrixmultiplikation ist nicht kommutativ: $A \cdot B \neq B \cdot A$

Beispiel:

verschieden!

(für $m=n=r$)

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 10 & 9 \end{pmatrix}$$

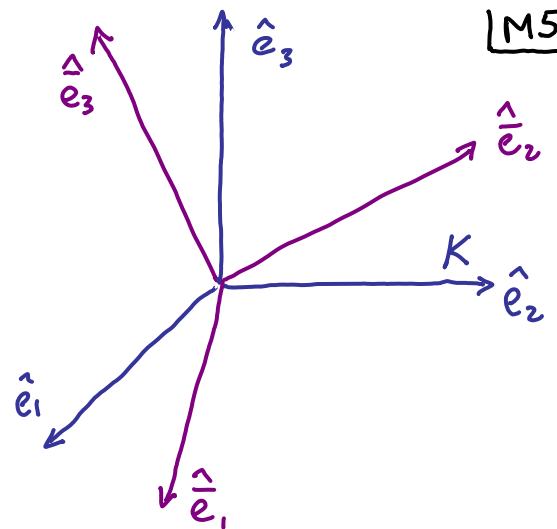
$$B \cdot A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 6 & 9 \end{pmatrix}$$

Koordinatentransformationen - Drehungen

$K, \bar{K}$ seien zwei Koordinatensysteme mit gleichem Ursprung, und orthonormalen Basisvektoren

$$\hat{e}_1, \hat{e}_2, \hat{e}_3, \text{ oder } \hat{\bar{e}}_1, \hat{\bar{e}}_2, \hat{\bar{e}}_3$$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \quad \hat{\bar{e}}_i \cdot \hat{\bar{e}}_j = \delta_{ij} \quad (1)$$



M5

Wir suchen zunächst

$\hat{\bar{e}}_j$, dargestellt in K :

$$\hat{\bar{e}}_j = \hat{e}_j$$

($\sum_m$ implizit)
(2)

"Drehmatrix" oder "Rotationsmatrix" (D_{jm}) ist nicht symmetrisch:

nicht gleich

$$D_{jm} = \hat{\bar{e}}_j \cdot \hat{e}_m =$$

$$D_{mj} = \hat{e}_m \cdot \hat{\bar{e}}_j =$$

Winkel von j-Achse in $\bar{K}$ mit m-Achse in K

nicht gleich!

(4)

Beispiel (2D):

M6

$$\hat{e}_1 = \hat{e}_1 \quad \hat{e}_2$$

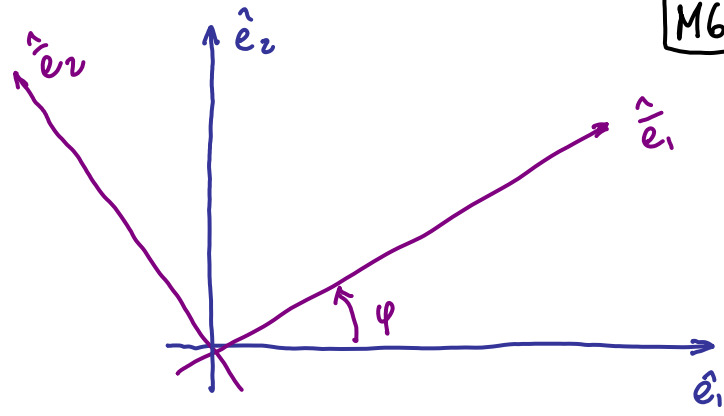
$$\hat{e}_2 = \hat{e}_1 \quad \hat{e}_2$$

$$D_{11} = \hat{e}_1 \cdot \hat{e}_1 =$$

$$D_{12} = \hat{e}_1 \cdot \hat{e}_2 =$$

$$D_{21} = \hat{e}_2 \cdot \hat{e}_1 =$$

$$D_{22} = \hat{e}_2 \cdot \hat{e}_2 =$$



Drehmatrix:

$$D_{jm} = \hat{e}_j \cdot \hat{e}_m = \begin{vmatrix} & \\ & \end{vmatrix}_{jm}$$

Ortsvektor $\vec{r}$ ist geometrische Größe
 $\Rightarrow$ unabhängig vom Koordinatensystem

(1a) M7

(1b)

(1b)
links

$$(x_i \hat{e}_i)^{(1b)} = (x_m \hat{e}_m) \quad (2)$$

(5.3)

$\bar{x}_j = D_{jm} x_m$

explizit:

$$\begin{matrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{matrix} = \begin{matrix} D_{21} x_1 + D_{22} x_2 + D_{23} x_3 \\ D_{31} x_1 + D_{32} x_2 + D_{33} x_3 \end{matrix} \quad (3)$$

= Matrixmultiplikation von
 3×3 Drehmatrix mit (3×1) -Matrix
 Orts (Spalten)-Vektor. 3×3 Drehmatrix

Kurznotation:

(5)

D beschreibt Drehung v. K nach $\bar{K}$

Def: "Inverse Matrix"

D^{-1} zu D erfüllt:

$$D^{-1} \cdot D = D \cdot D^{-1} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

LM8

(1)

(7.5)

$$\bar{x} = D \cdot x$$

(2)

$$(D_{ij}^{-1}) \bar{x}_j = x_i$$

(3)

D^{-1} beschreibt
Rückdrehung
von $\bar{K}$ nach K

(8.2)

$$(x_m \hat{e}_m) = (\bar{x}_j \hat{e}_j)$$

(4)

$$x_i \stackrel{(4)}{=} (D_{ij}^T) \bar{x}_j \stackrel{(3)}{=}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}$$

(5)

vergleiche mit (6.3)!

$$D^{-1} = D^T = (D_{ij}^{-1}) = (D_{ji})$$

Wichtig!!

(6)

Orthogonalität der
Basisvektoren impliziert:

$$\delta_{ij} = \hat{e}_i \cdot \hat{e}_j \stackrel{(5.2)}{=} (\hat{e}_i \cdot \hat{e}_j) = (D_{im} \hat{e}_m) \cdot (D_{jn} \hat{e}_n)$$

LM9

(1)

Zeilenvektoren von D
sind orthonormal!

$$\delta_{ij} = (\text{Zeilenvektor von } D) \cdot (\text{Zeilenvektor von } D)$$

(2)

Beispiel v. Seite 6:

$$D = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix};$$

$$\delta_{11} = () \cdot () = c^2 + s^2 = 1 \quad \checkmark$$

$$\delta_{12} = () \cdot () = cs - sc = 0 \quad \checkmark$$

Analoges gilt für Spaltenvektoren:

D^{-1} (5.2):

$$D_{jn}^{-1} \hat{e}_n = \hat{e}_j$$

(3)

$$\delta_{ij} = \hat{e}_i \cdot \hat{e}_j \stackrel{(3)}{=} (D_{in}^{-1} \hat{e}_n) \cdot (D_{jm}^{-1} \hat{e}_m) = \underbrace{(D_{in}^{-1} \hat{e}_n) \cdot (D_{jm}^{-1} \hat{e}_m)}_{\delta_{nm}}$$

(8.6)

(4)

Spaltenvektoren von D
sind orthonormal!

$$\delta_{ij} = (\text{Spaltenvektor von } D) \cdot (\text{Spaltenvektor von } D) \quad (5)$$

Beispiel v. Seite 6:

$$D = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix};$$

$$\delta_{11} = (c - s) \cdot (c - s) = c^2 + s^2 = 1 \quad \checkmark$$

$$\delta_{12} = (c - s) \cdot (s - c) = cs - sc = 0 \quad \checkmark$$