Zusammenfassung: Allgemeine Koordinatentransform. in 3D — 08-29.11.05 — F47

Def. der Transf: $x_i = x_i(y_j)$, $i = 1,2,3$; $j = 1,2,3$ (1)

Cartesisch — krummlinig

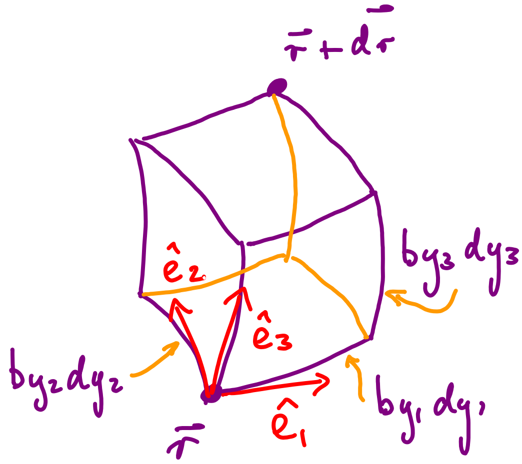
Ortsvektor: $\vec{r} = \sum_i x_i \hat{e}_i$ (2)

Normierungsfaktor

Lokales Dreibein: $\hat{e}_{y_i} \equiv \frac{\partial \vec{r}}{\partial y_i} \frac{1}{b_{y_i}}$, mit $b_{y_i} \equiv \left| \frac{\partial \vec{r}}{\partial y_i} \right|$ $i = 1,2,3$ (3)

Wegelement: $d\vec{r} = \sum_i \frac{\partial \vec{r}}{\partial y_i} dy_i$ (4)

$\overset{(3)}{=} \sum_i \underbrace{b_{y_i} dy_i}_{\text{Dimension: Länge}} \hat{e}_{y_i}$ (5)



Volumenelement: $dV = dx_1 dx_2 dx_3$

$= b_{y_1} b_{y_2} b_{y_3} dy_1 dy_2 dy_3$ (6)

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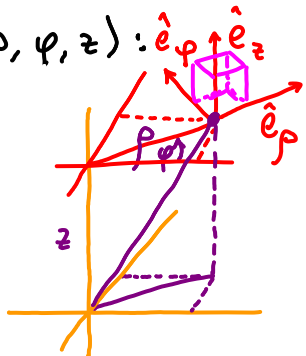
3D Polarkoordinaten: (ρ, φ, z): $\hat{e}_\varphi$, $\hat{e}_z$

$x = \rho \cos\varphi$,

$y = \rho \sin\varphi$,

$z = z$



$\vec{r} = \rho\cos\varphi\, \hat{e}_x + \rho\sin\varphi\, \hat{e}_y + z\, \hat{e}_z$

$\hat{e}_\rho = \cos\varphi\, \hat{e}_x + \sin\, \hat{e}_y$

$\hat{e}_\varphi = -\sin\varphi\, \hat{e}_x + \cos\, \hat{e}_y$

$\hat{e}_z = \hat{e}_z$

$b_\rho = 1$, $\underline{b_\varphi = \rho}$, $b_z = 1$

$d\vec{r} = d\rho\, \hat{e}_\rho + \rho\, d\varphi\, \hat{e}_\varphi + dz\, \hat{e}_z$

$dV = \rho\, d\rho\, d\varphi\, dz$

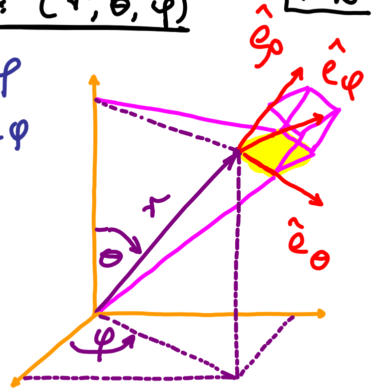
Kugelkoordinaten: (r, θ, φ)

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r\cos\theta$

$\vec{r} = r\, \hat{e}_r$



$\hat{e}_r = \sin\theta\cos\varphi\, \hat{e}_1 + \sin\theta\sin\varphi\, \hat{e}_2 + \cos\theta\, \hat{e}_3$

$\hat{e}_\theta = \cos\theta\cos\varphi\, \hat{e}_1 + \cos\theta\sin\varphi\, \hat{e}_2 - \sin\theta\, \hat{e}_3$

$\hat{e}_\varphi = -\sin\theta\sin\varphi\, \hat{e}_1 + \sin\theta\cos\varphi\, \hat{e}_2$

$b_r = 1$, $\underline{b_\theta = r}$, $\underline{b_\varphi = r\sin\theta}$

$d\vec{r} = dr\, \hat{e}_r + r\, d\theta\, \hat{e}_\theta + r\sin\theta\, \hat{e}_\varphi$

$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

David Griffiths, "Introduction to Electrodynamics", 3rd Ed., 1999

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$t(x,y,z)$ =: skalares Feld

$\vec{v}(x,y,z)$ =: Vektorfeld

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Gradient in Krummlinigen Koordinatensystemen

$$s = s(x_i) = s(x_i(y_j))$$

Def. v. Gradient (in Cartesischen KS):

$$\vec\nabla s \equiv \sum_i \hat e_i \frac{\partial s}{\partial x_i} = \sum_j \hat e_{y_j} (\vec\nabla s)_{y_j} \qquad (1)$$

$$\frac{\partial \vec r}{\partial y_j}\frac{1}{b_{y_j}} \quad (44.3) \qquad \delta_{ik} = \hat e_k \cdot \hat e_i \qquad (2)$$

Komponente v. $\vec\nabla\varphi$ in $\hat e_{y_k}$-Richtung:

$$(\vec\nabla s)_{y_j} \equiv \hat e_{y_j} \cdot \vec\nabla s = \sum_{ki} \left(\frac{\partial (x_k \hat e_k)}{\partial y_j} \frac{1}{b_{y_j}} \right) \cdot \left(\hat e_i \frac{\partial s}{\partial x_i} \right)$$

$$= \frac{1}{b_{y_j}} \underbrace{\sum_i \frac{\partial s}{\partial x_i} \frac{\partial x_i}{\partial y_j}}_{\frac{\partial s}{\partial y_j}} = \frac{1}{b_{y_j}} \frac{\partial s}{\partial y_j} \qquad (3)$$

Kettenregel

Gradient in y_j-Koordinaten ausgedrückt:

$$\vec\nabla s \overset{(1)}{=} \sum_j \hat e_{y_j} \frac{1}{b_{y_j}} \frac{\partial}{\partial y_j} s \qquad (4)$$

(5)

$$\boxed{\vec\nabla = \sum_j \hat e_{y_j} \frac{1}{b_{y_j}} \partial_{y_j}} \qquad (5)$$

Zylinderkoordinaten: $b_\rho = 1$, $b_\varphi = \rho$, $b_z = 1$

$$\vec\nabla = \hat e_\rho \partial_\rho + \hat e_\varphi \frac{1}{\rho} \partial_\varphi + \hat e_z \partial_z \qquad (1)$$

(47.4)

Kugelkoordinaten: $b_r = 1$, $b_\theta = r$, $b_\varphi = r\sin\theta$

$$\vec\nabla = \hat e_r \partial_r + \hat e_\theta \frac{1}{r} \partial_\theta + \hat e_\varphi \frac{1}{r\sin\theta} \partial_\varphi \qquad (2)$$

Divergenz in krummlinigen Koordinaten

in Cartesischen Koordinaten
" Krummlinigen "

Betrachte Vektorfeld:

$$\vec v = v_i \hat e_i =: v_{y_j} \hat e_{y_j} \qquad v_{y_j} = \vec v \cdot \hat e_{y_j} \qquad (3)$$

Divergenz cartesisch:

$$\vec\nabla \cdot \vec v = (\hat e_i \partial_i) \cdot (v_j \hat e_j) = \partial_i v_i \qquad (4)$$

(47.5) (3)

krummlinig:

$$= \sum_{ij} \left(\frac{\hat e_{y_i}}{b_{y_i}} \partial_{y_i} \right) \cdot \left(v_{y_j} \hat e_{y_j} \right) \qquad (5)$$

Ortsabhängig!! ⇒ Abl. wirkt auf Komponenten und auf Dreibein!!

Stunden später... (Siehe Nolting)

$$\vec\nabla \cdot \vec v = \frac{1}{b_{y_1} b_{y_2} b_{y_3}} \left[\partial_{y_1}(v_{y_1} b_{y_2} b_{y_3}) + \partial_{y_2}(b_{y_1} v_{y_2} b_{y_3}) + \partial_{y_3}(b_{y_1} b_{y_2} v_{y_3}) \right] \qquad (6)$$

Geometrische Interpretation: (Griffiths, Intr. to Electrodynamics, 3rd Ed. Anhang A.4) F49

Wegelement:

$$\Delta\vec{r} = \sum_i b_{y_i} \Delta y_i \,\hat{e}_{y_i}$$

Flächenelemente:

$$\Delta A_{1,2} = \pm\left(b_{y_2}\Delta y_2\, b_{y_3}\Delta y_3\right)\hat{e}_{y_1}$$

$$\Delta A_{3,4} = \pm\left(b_{y_1}\Delta y_1\, b_{y_3}\Delta y_3\right)\hat{e}_{y_2}$$

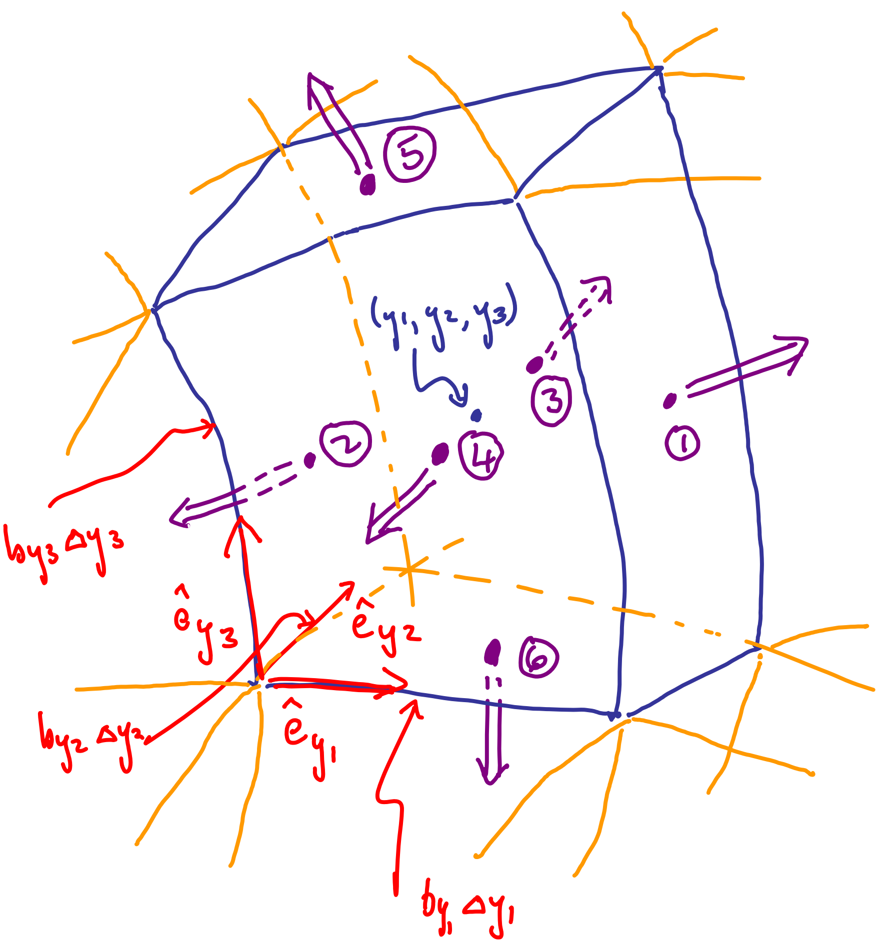
$$\Delta A_{5,6} = \pm\left(b_{y_1}\Delta y_1\, b_{y_2}\Delta y_2\right)\hat{e}_{y_3}$$

keine b's hier!

$$①,② = \left(y_1 \pm \frac{\Delta y_1}{2},\, y_2,\, y_3\right)$$

$$③,④ = \left(y_1,\, y_2 \pm \frac{\Delta y_2}{2},\, y_3\right)$$

$$⑤,⑥ = \left(y_1,\, y_2,\, y_3 \pm \frac{\Delta y_3}{2}\right)$$



Divergenz: $= \left(\dfrac{\text{Ausfluss}}{\text{Volumen } \Delta V}\right) = \dfrac{\sum_{i=1}^{6} (\vec{v}\cdot\Delta\vec{A})_{ⓘ}}{(b_{y_1}\Delta y_1)(b_{y_2}\Delta y_2)(b_{y_3}\Delta y_3)}$ (1) F50

$$= \frac{\left[\overbrace{(v_{y_1} b_{y_2} b_{y_3})_{①} - (v_{y_1} b_{y_2} b_{y_3})_{②}}^{\cancel{\Delta y_1}\,\partial_{y_1}(v_{y_1} b_{y_2} b_{y_3})}\right]\cancel{\Delta y_2}\,\cancel{\Delta y_3}}{(b_{y_1}\cancel{\Delta y_1})(b_{y_2}\cancel{\Delta y_2})(b_{y_3}\cancel{\Delta y_3})} \quad (2)$$

+ analog für ③ - ④ und ⑤ - ⑥

$$\boxed{\vec{\nabla}\cdot\vec{v} = \frac{1}{b_{y_1} b_{y_2} b_{y_3}}\left[\partial_{y_1}(v_{y_1} b_{y_2} b_{y_3}) + \partial_{y_2}(b_{y_1} v_{y_2} b_{y_3}) + \partial_{y_3}(b_{y_1} b_{y_2} v_{y_3})\right]} \quad (3)$$

Beispiele:

F51

Zylinderkoordinaten: $b_\rho = 1,\ b_\varphi = \rho,\ b_z = 1$:

$$\vec\nabla\cdot\vec v = \frac{1}{\rho}\Big[\partial_\rho(\overset{b_\varphi b_z}{\rho}\, v_\rho) + \partial_\varphi v_\varphi + \partial_z(\rho\, v_z)\Big] \qquad (1)$$

$$\boxed{\vec\nabla\cdot\vec v = \frac{1}{\rho}\partial_\rho(\rho v_\rho) + \frac{1}{\rho}\partial_\varphi v_\varphi + \partial_z v_z} \qquad (2)$$

Kugelkoordinaten: $b_r = 1,\ b_\theta = r,\ b_\varphi = r\sin\theta$

$$\vec\nabla\cdot\vec v = \frac{1}{r^2\sin\theta}\Big[\partial_r(r^2\sin\theta\, v_r) + \partial_\theta(r\sin\theta\, v_\theta) + \partial_\varphi(r\, v_\varphi)\Big] \qquad (3)$$

$$\boxed{\vec\nabla\cdot\vec v = \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\partial_\varphi v_\varphi} \qquad (4)$$

Anwendung:

Kugelkoord. $\qquad v_r = r,\ v_\theta = 0,\ v_\varphi = 0$

Sei $\vec v = \vec r = r\hat e_r \Rightarrow \vec\nabla\cdot\vec v = \frac{1}{r^2}\partial_r(r^2 r) = 3$ ✓

Sei $\vec v = \hat e_\varphi \Rightarrow \vec\nabla\cdot\vec v = 0 + 0 + \frac{1}{r\sin\theta}\partial_\varphi(1) = 0$

$v_r = 0,\ v_\varphi = 1,\ v_\theta = 0$

F52

Rotation:

$$\vec\nabla\times\vec v = \underbrace{(\hat e_i\,\partial_i)\times(v_j}_{\partial_i v_j\, \hat e_k\, \varepsilon_{ijk}}\,\hat e_j) = (\hat e_{y_i}\partial_{y_i})\times(v_{y_j}\hat e_{y_j}) \qquad (1)$$

Stunden später...

$$= \frac{1}{b_{y_2}b_{y_3}}\hat e_{y_1}\Big[\partial_{y_2}(b_{y_3}v_{y_3}) - \partial_{y_3}(b_{y_2}v_{y_2})\Big]$$

$$+ \frac{1}{b_{y_3}b_{y_1}}\hat e_{y_2}\Big[\partial_{y_3}(b_{y_1}v_{y_1}) - \partial_{y_1}(b_{y_3}v_{y_3})\Big] \qquad (2)$$

$$+ \frac{1}{b_{y_1}b_{y_2}}\hat e_{y_3}\Big[\partial_{y_1}(b_{y_2}v_{y_2}) - \partial_{y_2}(b_{y_1}v_{y_1})\Big]$$

[+ zyklisches Vertauschen von $1\to2\to3\to1$]

Zylinderkoordinaten: $b_\rho = 1,\ b_\varphi = \rho,\ b_z = 1$

$$\vec\nabla\times\vec v = \frac{1}{\rho}\hat e_\rho\Big[\partial_\varphi v_z - \partial_z(\rho v_\varphi)\Big] \quad\leftarrow\quad = \hat e_\rho\Big[\frac{1}{\rho}\partial_\varphi v_z - \partial_z v_\varphi\Big]$$

$$+ \hat e_\varphi\Big[\partial_z v_\rho - \partial_\rho v_z\Big] + \frac{1}{\rho}\hat e_z\Big[\partial_\rho(\rho v_\varphi) - \partial_\varphi v_\rho\Big]$$

Anwendung: Sei $\vec v = \hat e_\varphi$

$\vec\nabla\times\vec v =$

Kugelkoordinaten: Selber nachrechnen!

Geometrische Interpretation: (betrachte nur z-Komponente)

$$(\text{Rotation})_3 = \frac{\text{Zirkulation}}{\text{Fläche } \Delta A_3}$$

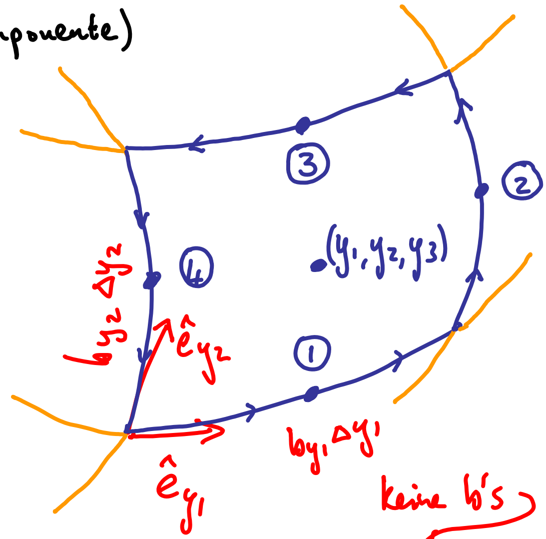
$$= \frac{1}{\Delta A}\sum_{i=1}^{4} (\vec{v}\cdot\Delta\vec{r})_{(i)}$$

$$= \frac{1}{b_{y_1} b_{y_2} \cancel{\Delta y_1}\,\cancel{\Delta y_2}}\Big\{\Big[\overbrace{(v_{y_1} b_{y_1})_{(1)} - (v_{y_1} b_{y_1})_{(3)}}^{-\cancel{\Delta y_2}\,\partial_2 (v_{y_1} b_{y_1})}\Big]\cancel{\Delta y_1}$$

$$\Big[\overbrace{(v_{y_2} b_{y_2})_{(2)} - (v_{y_2} b_{y_2})_{(4)}}^{\Delta y_1\,\partial_1 (v_{y_2} b_{y_2})}\Big]\Delta y_2\Big\}$$

$$(\vec{\nabla}\times\vec{v})_3 = \frac{1}{b_{y_1} b_{y_2}}\left[\partial_1 (b_{y_2} v_2) - \partial_2 (b_{y_1} v_1)\right]$$

analog für $(\vec{\nabla}\times\vec{v})_{1,2}$.



keine b's

$$(1),(3) = \left(y_1,\; y_2 \mp \frac{\Delta y_2}{2}\right)$$

$$(2),(4) = \left(y_1 \pm \frac{\Delta y_1}{2},\; y_2\right)$$

Wegelemente:

$$\Delta\vec{r}_{1,3} = \pm\, b_{y_1}\Delta y_1\,\hat{e}_{y_1}$$

$$\Delta\vec{r}_{2,4} = \pm\, b_{y_2}\Delta y_2\,\hat{e}_{y_2}$$

Flächenelement: $\Delta A_3 = b_{y_1}\Delta y_1\, b_{y_2}\Delta y_2$

Matrizen

Schöne Zusammenfassung: Nolting I

(hier ganz knapp; ausführlich $\Rightarrow$ lineare Algebra)

Definition einer

$m\times n$ Matrix

Anzahl Zeilen, Spalten

Spalten

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \vdots \\ \vdots & \vdots & A_{33} & \vdots \\ & & \ddots & \vdots \\ A_{m1} & \cdots & \cdots & A_{mn} \end{bmatrix} = (A_{ij}) \quad \begin{matrix} i = 1,\dots,m \\ j = 1,\dots,n \end{matrix} \qquad (1)$$

Matrix-Elemente

"Quadratische Matrix": falls $m = n$

(2)

"Null-Matrix": $A = 0 \Leftrightarrow A_{ij} = 0 \;\forall\, ij \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$

M2

"Diagonal-Matrix": (für $m=n$) $\quad d_{ij} = d_i \delta_{ij} \quad \forall i,j \quad \Longleftrightarrow \quad \begin{bmatrix} d_1 & & 0 \\ & d_2 \ddots & \\ 0 & & d_n \end{bmatrix}$ (1)

"Einheitsmatrix": (für $m=n$) $\quad E_{ij} = \delta_{ij} \quad \Longleftrightarrow \quad \begin{bmatrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{bmatrix} =: \mathbb{1}$ (2)

$A = (A_{ij})$ sei gegebene $m \times n$-Matrix;

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{m1} & \cdots & & A_{mn} \end{bmatrix}$$

erste Index: $1 \ldots m$

z.B. 2×3: $A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 4 & 6 \end{pmatrix}$

A_{23}

A_{32} gibt's nicht. (3)

"Transponierte Matrix" $n \times m$ ist $A^T = (A^T_{ij}) = (A_{ji})$

Vertausche Zeilen $\Leftrightarrow$ Spalten

n-Reihen
m-Spalten

$$A^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & & & \vdots \\ \vdots & & & \vdots \\ A_{1n} & \cdots & & A_{mn} \end{bmatrix} \qquad A_{mn} = A^T_{nm}$$

$A^T = \begin{pmatrix} 1 & 2 \\ -1 & 4 \\ 3 & 6 \end{pmatrix}$ (4)

$A^T_{32} = A_{23}$

M3

Rechenregeln: Seien $A = (A_{ij})$ und $B = (B_{ij})$ zwei $m \times n$-Matrizen (1)

Addition: $\quad C = A + B := (C_{ij}) \quad$ mit $C_{ij} = A_{ij} + B_{ij} \quad \forall i,j$

Skalare Multiplikation: $\quad \lambda A = (\lambda A_{ij}) \quad \forall i,j$ (2)

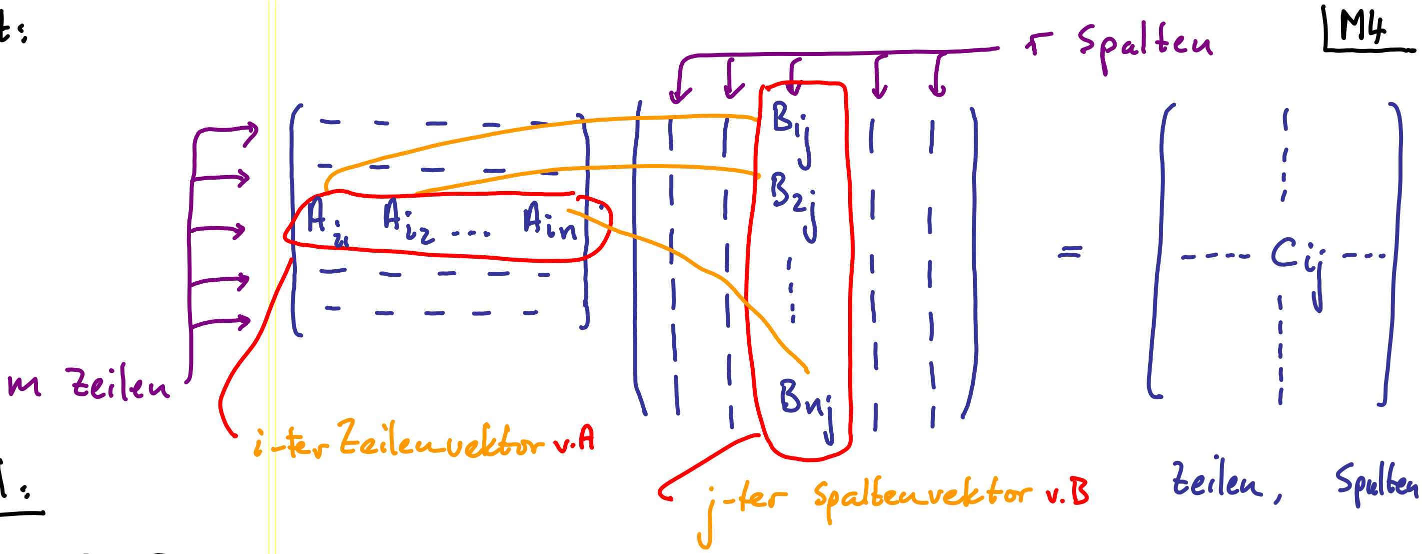
Beispiele: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}, \quad 3\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$ (3)

Seien $A = (A_{ij})$: $m \times n$ und $B = (B_{ij})$: $n \times r$

Matrixmultiplikation: $\quad C = A \cdot B = (C_{ij}), \quad$ mit $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \overset{\text{ESK}}{=} A_{ik} B_{kj}$ (4)

C_{ij} = "Skalarprodukt" aus i-tem "Zeilenvektor" (von A) und j-tem "Spaltenvektor" (von B)

Explizit:



Beispiel:

$m=3,\ n=2,\ r=2$

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

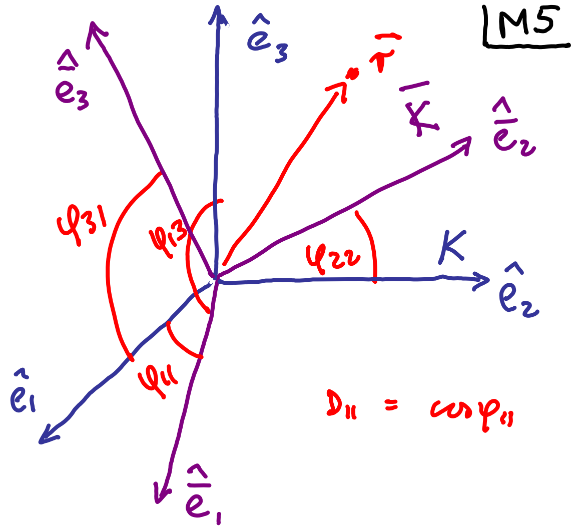
$$\begin{array}{c|c} & \begin{pmatrix} 0 & & & 1 \\ 2 & & & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1\cdot 0 + 2\cdot 2 & 1\cdot 1 + 2\cdot 1 \\ 4\cdot 0 + 5\cdot 2 & 4\cdot 1 + 5\cdot 1 \\ 1\cdot 0 + 0\cdot 2 & 1\cdot 1 + 0\cdot 1 \end{pmatrix} \end{array} = \begin{pmatrix} 4 & 3 \\ 10 & 9 \\ 0 & 1 \end{pmatrix}$$

Allgemein gilt: Matrixmultiplikation ist nicht kommutativ: $A\cdot B \neq B\cdot A$ (für $m=n=r$)

Beispiel: verschieden!

$$A\cdot B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 10 & 9 \end{pmatrix} \qquad B\cdot A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 6 & 9 \end{pmatrix}$$

Koordinatentransformationen – Drehungen



$K,\ \bar{K}$ seien zwei Koordinatensysteme mit gleichem Ursprung, und orthonormalen Basisvektoren

$$\hat{e}_1, \hat{e}_2, \hat{e}_3, \text{ oder } \hat{\bar{e}}_1, \hat{\bar{e}}_2, \hat{\bar{e}}_3$$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \qquad \hat{\bar{e}}_i \cdot \hat{\bar{e}}_j = \delta_{ij} \qquad (1)$$

Wir suchen zunächst $\hat{\bar{e}}_j$, dargestellt in K:

$$\hat{\bar{e}}_j = \underbrace{\left(\hat{\bar{e}}_j \cdot \hat{e}_m\right)}_{D_{jm}} \hat{e}_m =: D_{jm}\, \hat{e}_m \qquad \left(\textstyle\sum_m \text{ implizit}\right) \qquad (2)$$

"Drehmatrix" oder "Rotationsmatrix" (D_{jm}) ist nicht symmetrisch:

$$D_{jm} = \hat{\bar{e}}_j \cdot \hat{e}_m = \cos\varphi_{jm} \qquad (3)$$

Winkel von j-Achse in $\bar{K}$ mit m-Achse in K

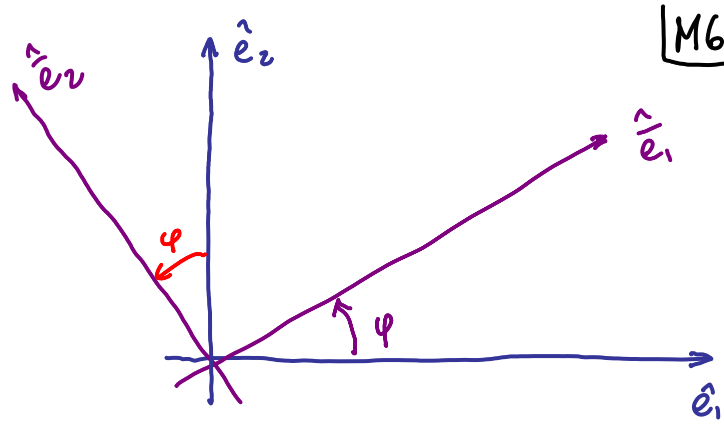
nicht gleich

$$D_{mj} = \hat{\bar{e}}_m \cdot \hat{e}_j = \cos\varphi_{mj} \qquad (4)$$

nicht gleich!

M6

Beispiel (2D):



$$\hat{\bar e}_1 = \cos\varphi \; \hat e_1 + \sin\varphi \; \hat e_2$$

$$\hat{\bar e}_2 = -\sin\varphi \; \hat e_1 + \cos\varphi \; \hat e_2$$

$$D_{11} = \hat{\bar e}_1 \cdot \hat e_1 = \cos\varphi \qquad D_{12} = \hat{\bar e}_1 \cdot \hat e_2 = \sin\varphi$$

$$D_{21} = \hat{\bar e}_2 \cdot \hat e_1 = -\sin\varphi \qquad D_{22} = \hat{\bar e}_2 \cdot \hat e_2 = \cos\varphi$$

Drehmatrix:
$$D_{jm} = \hat{\bar e}_j \cdot \hat e_m = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}_{jm}$$

$$D = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

M7

Ortsvektor $\vec r$ ist geometrische Größe
$\Rightarrow$ unabhängig vom Koordinatensystem

$$\vec r(\bar K) = \vec r(K) \qquad \text{(1a)}$$

$$\bar x_i \hat{\bar e}_i = x_m \hat e_m \qquad \text{(1b)}$$

$\hat{\bar e}_j \cdot$ (1b) ; links

$$\bar x_j = \hat{\bar e}_j \cdot \underbrace{(\bar x_i \hat{\bar e}_i)}_{\delta_{ij}} \overset{(1b)}{=} \underbrace{\hat{\bar e}_j \cdot (x_m \hat e_m)}_{D_{jm}} = D_{jm} x_m \qquad (2)$$

(5.3)

$\boxed{\bar x_j = D_{jm} x_m}$

explizit:

$$\begin{pmatrix} \bar x_1 \\ \bar x_2 \\ \bar x_3 \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{pmatrix} D_{11} x_1 + D_{12} x_2 + D_{13} x_3 \\ D_{21} x_1 + D_{22} x_2 + D_{23} x_3 \\ D_{31} x_1 + D_{32} x_2 + D_{33} x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}}_{\equiv D = (D_{ij})} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (3)$$

(4)

3x3 Drehmatrix

= Matrixmultiplikation von
3x3 Drehmatrix mit $\underbrace{(3\times1)\text{-Matrix}}_{\text{Orts (Spalten)-Vektor.}}$

Kurznotation: $\boxed{\bar x \overset{(3)}{=} D \cdot x}$ (5) $\qquad$ $\boxed{D \text{ beschreibt Drehung v. } K \text{ nach } \bar K}$