Nachholklausur: Musterlösung   TI, SoSe 2007   8.10.07 [1.1]

1. Harmonisches Potential

(1a)

Potential: $$V(x,y) = \frac{m\omega^2}{2}(x^2+y^2) \qquad (1)$$

Zylinderkoordinaten: $x = \rho\cos\phi, \quad y = \rho\sin\phi,$

Lagrange-Funktion: $L = T - V$

$$\boxed{L = \frac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2) - \frac{m\omega^2}{2}\rho^2} \qquad (2)$$

(1b) [1.2]

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = \frac{\partial L}{\partial\phi}$$

$$m\frac{d}{dt}(\rho^2\dot\phi) = 0$$

$\Rightarrow$ Erhaltener Drehimpuls: $$L = m\rho^2\dot\phi \qquad (3)$$

$$\boxed{\dot\phi = \frac{L}{m\rho^2}} \qquad (4)$$

da $L \neq L(t)$: Energieerhaltung:

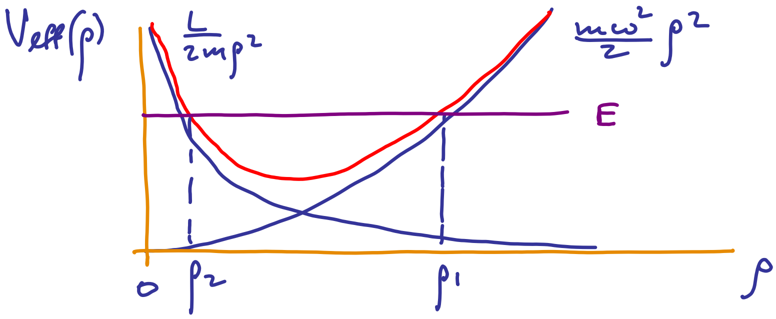
(1c)

$$E = \frac{1}{2}m(\dot\rho^2 + \rho^2\underbrace{\dot\phi^2}_{\overset{(4)}{=}\frac{L^2}{m^2\rho^4}}) + \frac{m\omega^2}{2}\rho^2 \qquad (5a)$$

$$= \frac{1}{2}m\dot\rho^2 + \underbrace{\frac{L^2}{2m\rho^2} + \frac{m\omega^2}{2}\rho^2}_{V_{eff}(\rho)} \qquad (5b)$$

1.3

mit $$\boxed{V_{eff}(\rho) = \frac{L^2}{2m\rho^2} + \frac{m\omega^2}{2}\rho^2} \quad (6)$$



(1d)

Umkehrpunkte $\rho = \rho_1$ oder ρ_2: wenn $\dot\rho = 0$, d.h.

$$E = V_{eff}(\rho) = \frac{L^2}{2m\rho^2} + \frac{1}{2}m\omega^2\rho^2 \quad (7)$$

Umformung: $\frac{2\rho^2}{m\omega^2} \times (10)$

$$\rho^4 - \frac{2E}{m\omega^2}\rho^2 + \frac{L^2}{m^2\omega^2} = 0$$

1.5

Zwei Lösungen: $$\left\{ \begin{matrix} \rho_1^2 \\ \rho_2^2 \end{matrix} \right\} = \frac{E}{m\omega^2} \pm \sqrt{\frac{E^2}{m^2\omega^4} - \frac{L^2}{m^2\omega^2}}$$

$$= \underline{\frac{E}{m\omega^2}\left[1 \pm \sqrt{1 - \frac{L^2\omega^2}{E^2}} \right]} \quad (8)$$

(1e) Für Kreisbahn ist $\rho_1 = \rho_2 = \sqrt{\frac{E}{m\omega^2}} \equiv \rho_0$ (9)

$$\overset{(8)}{\Longrightarrow} \sqrt{1 - \frac{L^2\omega^2}{E^2}} = 0 \Longrightarrow \boxed{E/L = \omega} \quad (10)$$

Aber: $$L \overset{(3)}{=} m\rho_0^2\dot\phi \quad (11)$$

$$\Longrightarrow \boxed{\dot\phi} \overset{(11)}{=} \frac{L}{m\rho_0^2} \overset{(9)}{=} \frac{L}{m}\left(\frac{m\omega^2}{E}\right) \overset{(10)}{=} \boxed{\omega} \quad (12)$$

(1f) Bonus: Radiale Kraft $F_\rho = -\frac{\partial V}{\partial \rho} = -m\omega^2\rho_0$ 1.5

Zentrifugalkraft $F_c = \frac{m v^2}{\rho_0} = m\rho_0\dot\phi^2$

denn: $v = \rho_0\dot\phi$

$|F_\rho| = |F_c| \Longrightarrow m\rho_0\dot\phi^2 = m\omega^2\rho_0$

$$\Longrightarrow \boxed{\dot\phi = \omega} \qquad (13)$$

Eingesetzt in: (5a), mit $\dot\rho = 0$

$$E \overset{(5a)}{=} \frac{1}{2}m\underbrace{v^2}_{=\rho_0^2\dot\phi^2} + \frac{1}{2}m\omega^2\rho_0^2 = m\omega^2\rho_0^2$$

$$L \overset{(3)}{=} m\rho_0^2\dot\phi \overset{(13)}{=} m\rho_0^2\omega$$

$\Longrightarrow E/L = \omega$ ✓ konsistent mit (10).

2. Kanonische Transformation: 2.1

Sei ω, α = konst.

$$H(q,p) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \alpha p q \qquad (1)$$

(2a)

$$\dot q = \frac{\partial H}{\partial p} = p + \alpha q \qquad (2a)$$

$$\dot p = -\frac{\partial H}{\partial q} = -\omega^2 q - \alpha p \qquad (2b)$$

(2b) $\frac{d}{dt}$(2a):

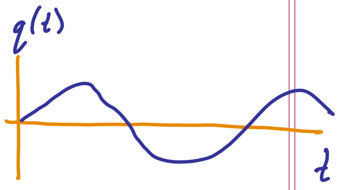
$$\ddot q = \dot p + \alpha\dot q \overset{(2b)}{=} (-\omega^2 q - \alpha p) + \alpha\dot q$$

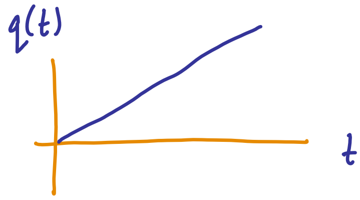
$$= -\omega^2 q + \alpha\dot q - \alpha[\dot q - \alpha q] \quad (2a)$$

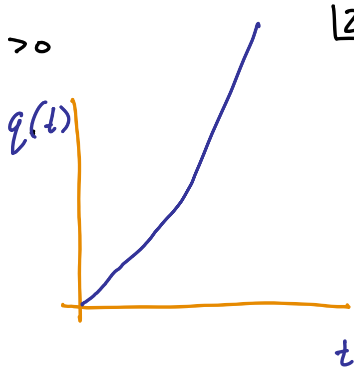
$$= -(\omega^2 - \alpha^2)q$$

$$\boxed{\ddot q + \Omega^2 q = 0} \quad (3) \qquad \boxed{\Omega^2 = \omega^2 - \alpha^2} \quad (4)$$

(2c) Anfangsbedingungen: $q(0) = 0, \quad \dot{q}(t) = v_0 > 0$







(i) $\Omega^2 > 0$

Lösung (nicht verlangt)

$$q(t) = \frac{v_0}{\Omega} \sin \Omega t$$

$$\Omega = \sqrt{\omega^2 - \alpha^2} > 0$$

(ii) $\Omega^2 = 0$

$$q(t) = v_0 t$$

(iii) $\Omega^2 < 0$

$$q(t) = \frac{v_0}{\tilde{\Omega}} \sinh \tilde{\Omega} t$$

$$\tilde{\Omega} = \sqrt{\alpha^2 - \omega^2} > 0$$

(2d) Erzeugende für kanonische Transformation:

$$F_2(q, P) = qP - \frac{A}{2} q^2 \tag{5}$$

Neue Koordinate: $$Q = \frac{\partial F_2}{\partial P} = q \tag{6a}$$

Alter Impuls: $$p = \frac{\partial F_2}{\partial q} = P - Aq \tag{6b}$$

Also:

$$\boxed{\begin{aligned} q(Q,P) &\overset{(6a)}{=} Q \qquad (7a) \\ p(Q,P) &\overset{(6b)}{=} P - AQ \qquad (7b) \end{aligned}}$$

Transformierte Hamiltonfunktion:

$$K(Q,P) = H(q,p) + \underbrace{\frac{\partial F_2}{\partial t}}_{=0 \text{ hier}}$$

$$\overset{(1)}{=} \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \alpha p q$$

$$\overset{(10)}{=} \frac{1}{2}(P - AQ)^2 + \frac{\omega^2 Q^2}{2} + \alpha(P - AQ)Q$$

$$= \frac{1}{2} P^2 + \frac{1}{2}(\omega^2 + A^2 - \alpha A) Q^2$$

$$+ PQ(-A + \alpha)$$

$\Rightarrow$ wähle $A = \alpha$, dann (8)

$$\boxed{K(Q,P) = \frac{1}{2} P^2 + \frac{1}{2}(\omega^2 - \alpha^2) Q^2} \qquad (9)$$

Kanonische Bewegungsgl.:

$$\dot{Q} = \frac{\partial K}{\partial P} \overset{(9)}{=} P \qquad (10a)$$

$$\dot{P} = -\frac{\partial K}{\partial Q} \overset{(9)}{=} -\frac{1}{2}(\omega^2 - \alpha^2) Q \qquad (10b)$$

$$\Rightarrow \boxed{\ddot{Q} \overset{(10a)}{=} \dot{P} \overset{(10b)}{=} -\frac{1}{2}(\omega^2 - \alpha^2) Q}$$

konsistent mit (4), ✓ wie erwartet, denn $q = Q$.

Check: Nicht verlangt:

Sind ursprüngliche Bewegungsgleichungen für p, q erfüllt, falls die für P, Q erfüllt sind?

$$\dot{q} \overset{?}{=} p + \alpha q \qquad\qquad \dot{p} = -\omega^2 q - \alpha p$$

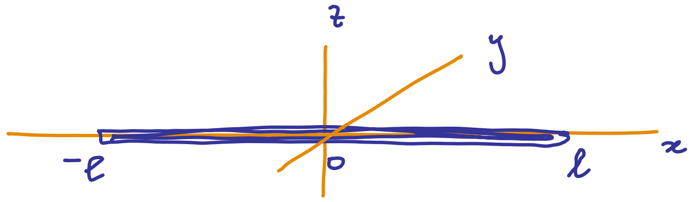
$$\dot{Q} = (P - \alpha Q) + \alpha Q \overset{\checkmark}{=} P \qquad\qquad (\dot{P} - \cancel{\alpha \dot{Q}}) = -\omega^2 Q - \alpha(P - \cancel{\alpha Q})$$

$$\dot{P} = -(\omega^2 - \alpha^2) Q \ \checkmark.$$

3. Stab-Feder System

3.1

(3a)

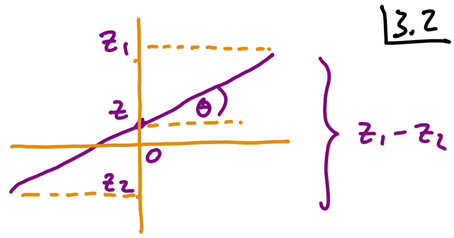


1-Dimensionale Massendichte:
$$\rho(x) = \frac{m}{2\ell}\,\Theta(\ell - |x|) \qquad (1)$$

Drehmoment:
$$I = \frac{m}{2\ell}\int_{-\ell}^{\ell} dx\, x^2 = \frac{m}{2\ell}\,\frac{1}{3}\,x^3\Big|_{-\ell}^{\ell}$$
$$= \frac{m}{2\ell}\,\frac{1}{3}\,2\,\ell^3$$
$$\boxed{I = \frac{1}{3} m \ell^2} \qquad (2)$$

3.2

(3b)
$$z = \frac{z_1 + z_2}{2}, \quad \sin\theta = \frac{z_1 - z_2}{2\ell}$$



$$\left.\begin{aligned} z_1 &= z + \ell\sin\theta \\ z_2 &= z - \ell\sin\theta \end{aligned}\right\} \quad (3)$$

Pot. Energie:
$$V(z_1, z_2) = \frac{k_1}{2} z_1^2 + \frac{k_2}{2} z_2^2$$
$$= \frac{k_1}{2}\left(z^2 + 2z\ell\sin\theta + \ell^2\sin^2\theta\right)$$
$$+ \frac{k_2}{2}\left(z^2 - 2z\ell\sin\theta + \ell^2\sin^2\theta\right) \qquad (4)$$

Linearisieren:
$\sin\theta \simeq \theta$
$$\simeq \frac{1}{2}(k_1 + k_2)\,z^2 + (k_1 - k_2)\,\ell\, z\,\theta$$
$$+ \frac{1}{2}(k_1 + k_2)\,\ell^2\theta^2 \qquad (5)$$

Sei:
$$\boxed{\begin{aligned} K &= k_1 + k_2 \\ k &= k_1 - k_2 \end{aligned}} \qquad \boxed{V = \frac{1}{2} K\left(z^2 + \ell^2\theta^2\right) + k\,\ell\, z\,\theta} \qquad (6)$$

(3c) $\quad T = T_{sp} + T_{Rot} = \boxed{\frac{1}{2} m \dot{z}^2 + \frac{1}{2} I \dot{\theta}^2 \quad I = \frac{1}{3} m \ell^2}$ $\quad$ 3.3 $\quad$ (7)

(3d) Lagrange: $L = T - V$

Bewegungsgleichungen: $\quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

für z: $\quad m \ddot{z} = -(K z + k \ell \theta)$

$$\frac{1}{3} m \ell^2 \ddot{\theta} = -(K \ell^2 \theta + k \ell z)$$

Matrix-Form: $\quad \boxed{\begin{pmatrix} m & 0 \\ 0 & \frac{1}{3} m \end{pmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{\theta} \end{pmatrix} = - \begin{pmatrix} K & k\ell \\ k/\ell & K \end{pmatrix} \begin{pmatrix} z \\ \theta \end{pmatrix}}$ $\quad$ (8)

mit $\quad K = k_1 + k_2, \quad k = k_1 - k_2$

(3e) $\quad$ 3.4

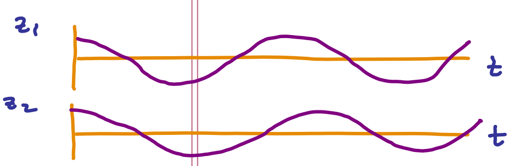
Charakteristisches Polynom: $\quad \boxed{(K - \omega^2 m)(K - \omega^2 \tfrac{1}{3} m) - k^2 = 0}$ $\quad$ (9)

(3f) Sei $k_1 = k_2 \Rightarrow k = 0$.

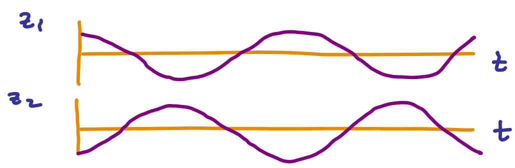
Schwerpunktsoszillationen: $\quad \omega_1 = \sqrt{K/m}, \quad \begin{pmatrix} z_0 \\ \theta_0 \end{pmatrix}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Winkeloszillationen: $\quad \omega_2 = \sqrt{\frac{3K}{m}}, \quad \begin{pmatrix} z_0 \\ \theta_0 \end{pmatrix}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(3g)



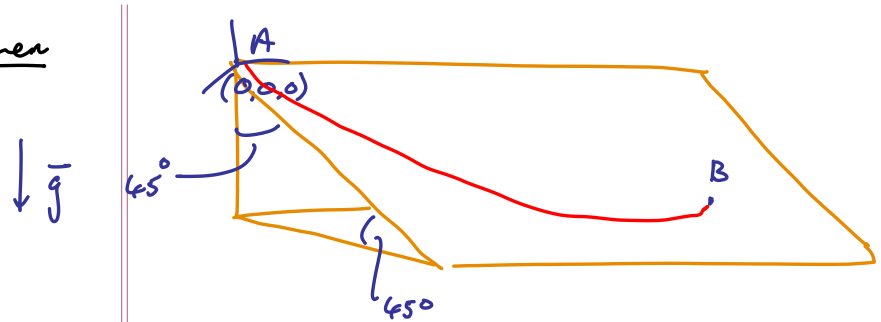
Mode 1



Mode 2

4. Ski-Rennen

4.1



(4a) Zwangsbedingung: $\boxed{-z = x \tan\theta = x}$ (1)

(4b) Energieerhaltung: $E = \frac{1}{2} m v^2(t) + mg(\underbrace{z(t) - z_A}_{}) \equiv 0$, mit $z_A \overset{!}{=} 0$

$$v = \sqrt{-2gz} = \boxed{\sqrt{2gx} = v} \quad (2)$$

(4c) Fahrzeit: $T = \int_A^B dt = \int_A^B ds \, \frac{1}{v}$

4.2

Aber: $ds = \sqrt{dx^2 + dy^2 + dz^2}$

mit $dz = -dx$

$$= \sqrt{2 dx^2 + dy^2} = dy \sqrt{1 + 2 \underbrace{(dx/dy)^2}_{\equiv x'}}$$

$$\Rightarrow \boxed{T = \int_0^{y_B} dy \, \underbrace{\frac{\sqrt{1 + 2x'^2}}{\sqrt{2gx}}}_{F(x, \frac{dx}{dy})}} \quad (3)$$

← keine y-Abhängigkeit

(4d) Erhaltene Größe: $c = F - x' \frac{\partial F}{\partial x'}$

$$= \sqrt{\frac{1 + 2x'^2}{2gx}} - x' \frac{2x'}{\sqrt{(1+2x'^2)\, 2gx}}$$

$$= \frac{1}{\sqrt{(1+2x'^2)\, 2gx}}$$

4.3

$$2x'^2+1 = \frac{1}{2gxc^2}$$

$$\boxed{\frac{dx}{dg} = x' = \pm\sqrt{\frac{1}{4gc^2x} - \frac{1}{2}}} \quad (4)$$

$$y = \int dy = \pm\int_0 dx \sqrt{\frac{4gc^2x}{1-2gc^2x}} + a$$

(4e) Substitution : $x = \frac{1}{2c^2g}\sin^2\varphi/2 = \frac{1}{4c^2g}(1-\cos\varphi)$

$$dx = \frac{1}{2c^2g}\sin\frac{\varphi}{2}\cos\varphi/_2\, d\varphi$$

$$y = \pm\int_0 \frac{d\varphi}{2c^2g}\sin\frac{\varphi}{2}\cancel{\cos\frac{\varphi}{2}}\;\frac{\sqrt{2}\,\sin\varphi/2}{\sqrt{1-\sin^2\varphi/2}} + a$$

4.4

$$= \pm\frac{1}{\sqrt{2}c^2g}\underbrace{\int d\varphi\,\sin^2\varphi/_2}_{\frac{1}{2}\left(\varphi/_2 - \underbrace{\cos\varphi/_2\sin\varphi/_2}_{\frac{1}{2}\sin\tilde{\varphi}}\right)} + a$$

$$\boxed{y = \pm\left(\frac{1}{\sqrt{2}c^2g}\right)\frac{1}{4}(\varphi - \sin\varphi) + a} \quad (5)$$

$x = 0 \Rightarrow \varphi = 0$ und $y = 0 \Rightarrow a = 0.$

(4f)

Also gilt:

$$x = \overbrace{\frac{1}{4c^2g}}^{r_x}(1-\cos\varphi(y)) = \boxed{r_x(1-\cos\varphi) = x}$$

$$y = \pm\underbrace{\left(\frac{1}{\sqrt{2}\,4c^2g}\right)}_{r_y}(\varphi - \sin\tilde{\varphi}) = \boxed{\pm r_y(\tilde{\varphi} - \cos\varphi) = y} \quad (6)$$

mit $\underline{r_x/r_y = \sqrt{2}}$

Nicht verlangt:

Durch Einsetzen der Endpunktwerte (x_B, y_B), werden φ_B und c festgelegt:

$$x_B = \frac{1}{4c^2 g}\left(1 - \cos\varphi_B\right)$$

$$y_B = \frac{1}{\sqrt{2}\, 4c^2 g}\left(\varphi_B - \sin\varphi_B\right)$$