Probeklausur : T1 , SoSe 07 – Musterlösung

Aufgabe 1 : Vektorkalkulus

1(a) $\phi(x,y,z) = xy^2$

$$\vec{A} = A_1\hat{x} + A_2\hat{y} + A_3\hat{z} := \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \vec{\nabla}\phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{pmatrix} = \begin{pmatrix} y^2 \\ 2xy \\ 0 \end{pmatrix}$$

1(b) $\vec{B} := B_1\hat{x} + B_2\hat{y} + B_3\hat{z} := \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$

$$= \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{pmatrix} \partial_y A_3 - \partial_z A_2 \\ \partial_z A_1 - \partial_x A_3 \\ \partial_x A_2 - \partial_y A_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2y - 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1(c) :

$$(\vec{\nabla}\times\vec{\nabla}\tilde{\phi})_i = \varepsilon_{ijk}\,\partial_j\partial_k\tilde{\phi} \quad (1)$$

$$= -\varepsilon_{ikj}\,\partial_j\partial_k\tilde{\phi} \quad \text{(Antisymmetrie v. } \varepsilon_{ijk}\text{)}$$

$$= -\varepsilon_{ijk}\,\partial_k\partial_j\tilde{\phi} \quad (\text{Umbenennung } j \leftrightarrow k)$$

$$= -\varepsilon_{ijk}\,\partial_j\partial_k\tilde{\phi} \quad (\partial_k\partial_j\tilde{\phi} = \partial_j\partial_k\tilde{\phi}) \quad (2)$$

$$(1) = (2) = -(1) \Rightarrow (1) = 0 \Rightarrow \vec{\nabla}\times\vec{\nabla}\tilde{\phi} = 0 . \quad \square$$

2. Erhaltungssätze

$$V = \tfrac{1}{2}k(\vec{r}_1 - \vec{r}_2)^2 = \tfrac{1}{2}k\left[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\right]$$

2(a)

$$\dot{\vec{p}}_1 = \vec{F}_1 = -\vec{\nabla}_1 V = k(\vec{r}_1 - \vec{r}_2) \quad (1a)$$

$$\dot{\vec{p}}_2 = \vec{F}_2 = -\vec{\nabla}_2 V = -k(\vec{r}_1 - \vec{r}_2) \quad (1b)$$

③

2(b) Gesamtimpuls: $\vec{P} = \vec{p}_1 + \vec{p}_2$

$$\frac{d}{dt}\vec{P} = \dot{\vec{P}} = \dot{\vec{p}}_1 + \dot{\vec{p}}_2 = \overset{(1a)}{k(\vec{r}_1 - \vec{r}_2)} - \overset{(1b)}{k(\vec{r}_1 - \vec{r}_2)} = 0 \qquad \square$$

2(c) Gesamtdrehimpuls: $\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$

$$\frac{d}{dt}\vec{L} = \dot{\vec{L}} = \left(\dot{\vec{r}}_1 \times \vec{p}_1 + \vec{r}_1 \times \dot{\vec{p}}_1\right) + \left(\dot{\vec{r}}_2 \times \vec{p}_2 + \vec{r}_2 \times \dot{\vec{p}}_2\right)$$

$$= \left[\overset{=0}{\cancel{\dot{\vec{r}}_1 \times m_1 \dot{\vec{r}}_1}} + \vec{r}_1 \times k(\overset{=0}{\cancel{\vec{r}_1}} - \vec{r}_2)\right] + \left[\overset{=0}{\cancel{\dot{\vec{r}}_2 \times m_2 \dot{\vec{r}}_2}} + \vec{r}_2 \times (-k)(\vec{r}_1 - \overset{=0}{\cancel{\vec{r}_2}})\right]$$

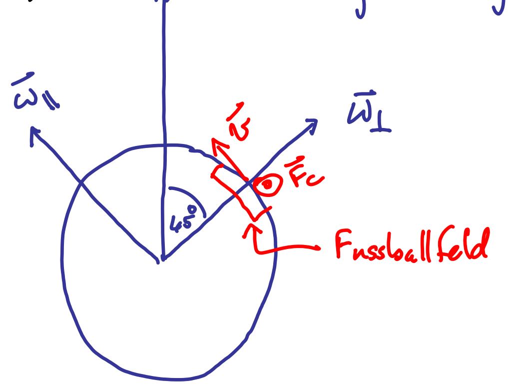
$$= -k\left[\vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_1\right] = 0 \qquad \square.$$

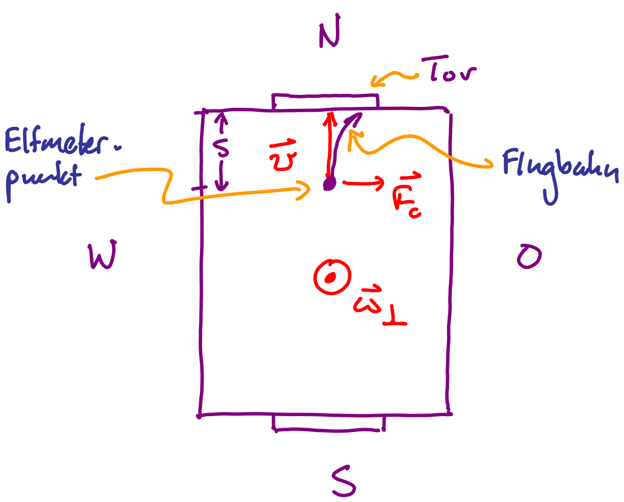
④

3. Coriolis-Kraft beim Elfmeterschuss

(3a) $\vec{\omega}$ = Winkelgeschwindigkeit der Erde

Blick von oben auf das Feld:





Coriolis-Kraft: $\vec{F}_c = 2m\,\vec{v} \times \vec{\omega}_{\perp}$ (1)

(3b): Nehme Coriolis-Beschleunigung als konstant an: $a_c = |\vec{a}_c| = 2\,v\,\omega_{\perp}$

Flugdauer vom Elfmeterpunkt zum Tor: $t = \frac{s}{v}$ (2)

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$$\text{Auslenkung} \sim \frac{1}{2} a_c t^2 \overset{(1,2)}{=} \frac{1}{2}(2v\omega_\perp)\left(\frac{s}{v}\right)^2 = \frac{\omega_\perp s^2}{v} \qquad (3)$$

$$\omega_\perp = \frac{2\pi}{1\ \text{Tag}} = \frac{\overset{\simeq 6}{2\pi}}{24 \times 60 \times 60\ \text{s}} \simeq \frac{1}{\underset{\simeq 150}{4 \times 6 \times 6} \times 10^2\ \text{s}} \simeq \frac{1}{15000\ \text{s}}$$

$$\text{Auslenkung} \simeq \frac{(11\ \text{m})^2}{(1.5 \times 10^4\ \text{s})(3 \times 10^1\ \text{m.s}^{-1})} \simeq \frac{10^2}{5 \times 10^5} \simeq 2.10^{-4}\ \text{m}$$

Größenordnung der Auslenkung: $\boxed{10^{-4}\ \text{m}}$.

Jede Antwort zwischen 10^{-5} und 10^{-3} ist akzeptabel.

4. Eigenschwingungen

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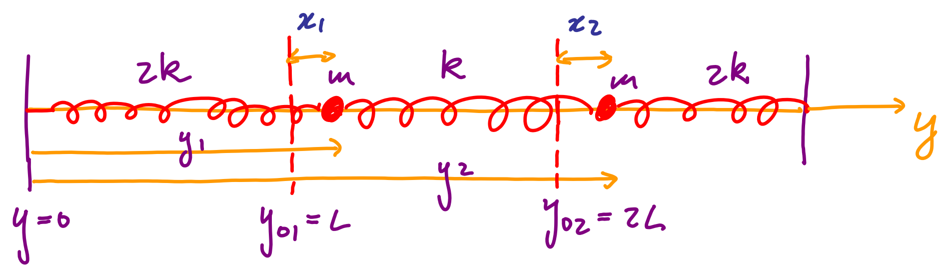


4(a) $$V = \frac{1}{2}(2k)(y_1 - y_{01})^2 + \frac{1}{2}k(y_2 - y_1 - L)^2 + \frac{1}{2}(2k)(y_2 - y_{02})^2 \qquad (1)$$

4(b) $$x_i = y_i - y_{0i} \qquad (2)$$

(2) in (1): $$V = \frac{1}{2}k\left[2x_1^2 + (x_1 - x_2)^2 + 2x_2^2\right]$$

$$= \frac{1}{2}k\left[3x_1^2 - 2x_1x_2 + 3x_2^2\right] \qquad (3)$$

⑦

$$V = \frac{1}{2} (x_1, x_2) \underbrace{k\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}}_{\hat{V}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} \sum_{ij} x_i \hat{V}_{ij} x_j \qquad (4)$$

4(c) $$\hat{T}_{ij} \ddot{x}_j =: m_i \ddot{x}_i = F_i = -\frac{\partial}{\partial x_i} V = -\hat{V}_{ij} x_j$$

mit $$\hat{T}_{ij} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}_{ij} = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{denn } m_1 = m_2 = m. \qquad (5)$$

$$\Rightarrow \quad \left(\hat{T}_{ij} \frac{d^2}{dt^2} + \hat{V}_{ij} \right) x_j = 0 \qquad (6)$$

4(d) Ansatz: $x_j(t) = a_j e^{i\omega t}$, eingesetzt in (6):

$$\left(-\hat{T}_{ij} \omega^2 + \hat{V}_{ij} \right) a_j = 0 \quad . \qquad (7)$$

⑧

Schreibe $\omega = \sqrt{k/m}\, \nu$, ν = dimensionslos. (8)

Eigenwertproblem: $$\left[-\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} + k \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \right] \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 .$$

oder $$\begin{bmatrix} 3-\nu^2 & -1 \\ -1 & 3-\nu^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 . \qquad (9)$$

Charakteristisches Polynom: $$(3-\nu^2)^2 - 1 = 0$$

$$\nu^4 - 6\nu^2 + 9 - 1 = 0$$

$$(\nu^2 - 2)(\nu^2 - 4) = 0$$

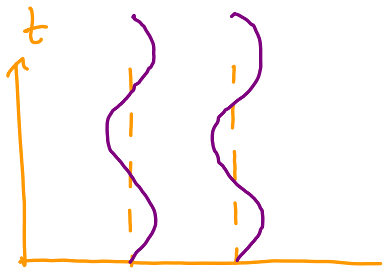
Eigenfrequenzen: $$\boxed{\nu^{(1)} = \sqrt{2}, \qquad \nu^{(2)} = 2} \qquad (10)$$

Eigenmoden:

$\alpha = 1$: $\nu^{(1)} = \sqrt{2}$ in (9): $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1^{(1)} \\ a_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

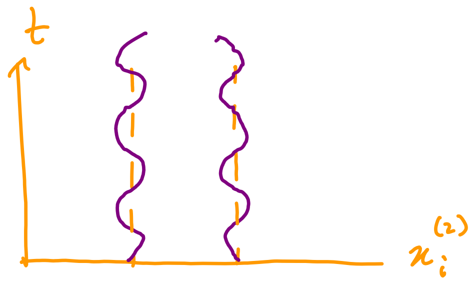
$\alpha = 2$: $\nu^{(2)} = 2$ in (9): $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_1^{(2)} \\ a_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\alpha = 1$



Beachte:

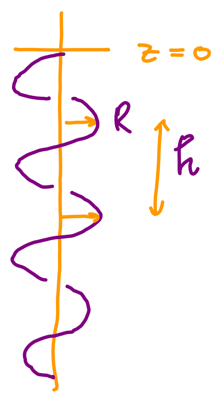
$\nu^{(1)} = \frac{1}{\sqrt{2}} \nu^{(2)}$



Schwingungen haben dieselbe Phase

Schwingungen sind um π phasenverschoben.

5. Schraubenbewegung



Zylinderkoordinaten:

$$x = \rho \cos\varphi \qquad y = \rho \sin\varphi \qquad z = z \tag{1}$$

5(a)

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{2}$$

$$= \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2 + \dot{z}^2) \tag{3}$$

5(b) Zwangsbedingung:

Radius fest: $\rho = R$ (4)

Höhe steigt linear mit φ: $z = h \frac{\varphi}{2\pi}$ [Winkel sei nicht auf das Intervall $[0, 2\pi)$ eingeschränkt!] (5)

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z sei verallg. Koordinate $\Rightarrow$ $\boxed{\begin{aligned} \rho(z) &= R = \text{konst} \\ \varphi(z) &= \frac{2\pi}{h} z \end{aligned}}$ (6) (7)

5(c) $$L(z,\dot z) = T - U$$

$$= \frac{1}{2} m \left[\rho^2 \dot\varphi^2 + \dot z^2\right] - mgz$$

$$= \frac{1}{2} m \left[\left(\frac{R\,2\pi}{h}\right)^2 + 1\right] \dot z^2 - mgz \quad (8)$$

Lagrange z:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot z}\right) = \frac{\partial L}{\partial z} \quad (9)$$

$$m\left[\left(\frac{R\,2\pi}{h}\right)^2 + 1\right] \ddot z = -mg \quad (10)$$

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$$\ddot z = -\tilde g \quad , \quad \tilde g = \underbrace{\frac{g}{\left(\frac{R\,2\pi}{h}\right)^2 + 1}}_{= \text{Effektives } g} \quad (11)$$

5(d) Lösung v. (11):

$$z(t) = -\frac{1}{2}\tilde g t^2 = \frac{-1}{\left(\frac{R\,2\pi}{h}\right)^2 + 1} \frac{1}{2}\tilde g t^2 \quad (12)$$

$$\varphi(t) \overset{(7)}{=} \frac{2\pi}{h} z(t) = \frac{-1}{R^2 \frac{2\pi}{h} + \frac{h}{2\pi}} \frac{1}{2}\tilde g t^2 \quad (13)$$

$$\rho(t) \overset{(6)}{=} R \quad (14)$$