Musterlösung zur Klausuraufgaben 2 & 3

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2. Durchhängende Hochspannungsleitung

a) $$\mathcal{U}[y(x)] = \int_{-x_0}^{x_0} \frac{m}{\ell} g\, y(x) \sqrt{1+[y'(x)]^2}\, dx$$

b) $$L[y(x)] = \int_{-x_0}^{x_0} \sqrt{1+[y'(x)]^2}\, dx = \ell$$

c) $$\begin{cases} \mathcal{U}[y(x)] = \text{minimal} \\ L[y(x)] = \ell \end{cases}$$

$$\mathcal{U}[y(x)] + \lambda\left(L[y(x)] - \ell\right) = \text{minimal}$$

$$\int_{-x_0}^{x_0} \left\{ \frac{mg}{\ell} y(x)\sqrt{1+[y'(x)]^2} + \lambda\left(\sqrt{1+[y'(x)]^2} - \frac{\ell}{2x_0}\right)\right\} dx = \text{minimal}$$

$$\delta \int_{-x_0}^{x_0} F(y,y',x)\, dx = 0$$

$$F(y,y',x) = \left(\frac{mg}{\ell} y + \lambda\right)\sqrt{1+y'^2} - \lambda \frac{\ell}{2x_0}$$

d) $$F - y'\frac{\partial F}{\partial y'} = C$$

$$\frac{\partial F}{\partial y'} = \left(\frac{mg}{\ell} y + \lambda\right)\frac{y'}{\sqrt{1+y'^2}}$$

$$F - y'\frac{\partial F}{\partial y'} = \left(\frac{mg}{\ell}y + \lambda\right)\sqrt{1+y'^2} - \lambda\frac{\ell}{2x_0} -$$

$$- y'\left(\frac{mg}{\ell}y + \lambda\right)\frac{y'}{\sqrt{1+y'^2}} = C$$

$$\left(\frac{mg}{\ell}y + \lambda\right)\frac{1}{\sqrt{1+y'^2}} - \lambda\frac{\ell}{2x_0} = C$$

$$y' = \pm\sqrt{\left(\frac{\frac{mg}{\ell}y + \lambda}{C + \frac{\lambda\ell}{2x_0}}\right)^2 - 1}$$

e) $$\frac{dy}{dx} = \pm\sqrt{\left(\frac{\frac{mg}{\ell}y + \lambda}{C + \frac{\lambda\ell}{2x_0}}\right)^2 - 1}$$

$$\int \frac{dy}{\sqrt{\left(\frac{\frac{mg}{\ell}y + \lambda}{C + \frac{\lambda\ell}{2x_0}}\right)^2 - 1}} = \pm x + a$$

$$\frac{C + \frac{\lambda\ell}{2x_0}}{\frac{mg}{\ell}}\int\frac{dz}{\sqrt{z^2-1}} = \pm x + a, \qquad z = \frac{\frac{mg}{\ell}y + \lambda}{C + \frac{\lambda\ell}{2x_0}}$$

$$\operatorname{arccosh}\left(\frac{\frac{mg}{\ell}y + \lambda}{C + \frac{\lambda\ell}{2x_0}}\right) = \frac{\frac{mg}{\ell}(a \pm x)}{C + \frac{\lambda\ell}{2x_0}}$$

$$\frac{mg}{\ell}y + \lambda = \left(C + \frac{\lambda\ell}{2x_0}\right)\cosh\left(\frac{\frac{mg}{\ell}(a \pm x)}{C + \frac{\lambda\ell}{2x_0}}\right)$$

$$y = \frac{\ell}{mg}\left\{\left(c+\frac{\lambda\ell}{2x_0}\right)\cosh\left(\frac{\frac{mg}{\ell}(a\pm x)}{c+\frac{\lambda\ell}{2x_0}}\right)-\lambda\right\}$$

One could also absorb $\frac{\lambda\ell}{2x_0}$ into the c-constant. An equivalent solution is, therefore,

$$y = \frac{\ell}{mg}\left\{c\cosh\left(\frac{\frac{mg}{\ell}(a\pm x)}{c}\right)-\lambda\right\}$$

Furthermore, $\frac{\ell}{mg}c$ could be also taken into c.

$$y = c\cosh\left(\frac{a\pm x}{c}\right)-\frac{\lambda\ell}{mg}.$$

All three solutions are equivalent from point of view of solving differential equations.

f) $y(-x_0)=0$

$$y(x_0)=0$$

$$\int_{-x_0}^{x_0}\sqrt{1+[y'(x)]^2}\,dx = \ell$$

Facultative: Let's apply these conditions.

$$0 = c\cosh\left(\frac{a\pm x_0}{c}\right)-\frac{\lambda\ell}{mg}$$

$$0 = c\cosh\left(\frac{a\mp x_0}{c}\right)-\frac{\lambda\ell}{mg}$$

$$\int_{-x_0}^{x_0}\sqrt{1+\sinh^2\left(\frac{a\pm x}{c}\right)}\,dx = \ell$$

From the first two it follows that $a=0$. Then, we are left with:

$$0 = c\cosh\left(\frac{x_0}{c}\right)-\frac{\lambda\ell}{mg}$$

$$\int_{-x_0}^{x_0} \cosh\left(\frac{x}{c}\right) dx = l$$

The first equation expresses λ via c

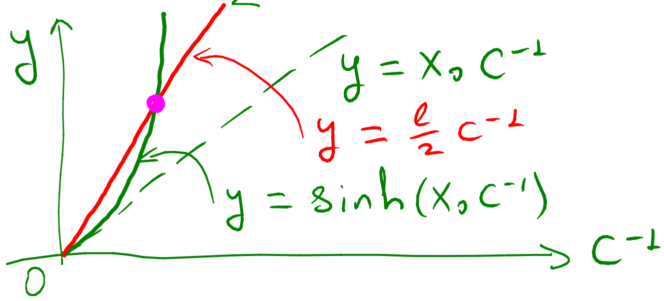
$$\lambda = \frac{mg}{l} c \cosh\left(\frac{x_0}{c}\right)$$

The second equation is for determining c

$$2c \sinh\left(\frac{x_0}{c}\right) = l$$

This equation is transcendental, but it has a finite solution for c, provided $x_0 < \frac{l}{2}$:

$$\boxed{\sinh\left(\frac{x_0}{c}\right) = \frac{l}{2c}}$$



$$\begin{cases} y(x) = c \cosh\left(\frac{x}{c}\right) - c \cosh \frac{x_0}{c} \\ \sinh\left(\frac{x_0}{c}\right) = \frac{l}{2c} \end{cases}$$

3. Überkippen eines Würfels

a) $$I_{xx}^{SP} = \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dz \frac{m}{a^3} (y^2 + z^2) =$$

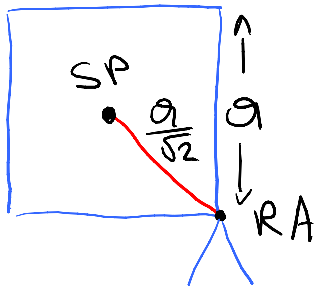
$$= \frac{m}{a^3}\left(a^2 \frac{1}{3} y^3 \Big|_{-\frac{a}{2}}^{\frac{a}{2}} + a^2 \frac{1}{3} z^3 \Big|_{-\frac{a}{2}}^{\frac{a}{2}}\right) = \frac{ma^2}{6}$$

$$I_{xy}^{SP} = -\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dz \frac{m}{a^3} xy = 0$$

$$I_{ij}^{SP} = \begin{pmatrix} \frac{ma^2}{6} & 0 & 0 \\ 0 & \frac{ma^2}{6} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{pmatrix}, \qquad I_{xx}^{SP} = I_{yy}^{SP} = I_{zz}^{SP} = \frac{ma^2}{6}$$

b) All main moments of inertia are equal to each other. Therefore, one can choose the direction of one main axis arbitrarily. The tensor is diagonal in any coordinate frame.

c) $I_{ij} = I_{ij}^{SP} + m\left(r^2 \delta_{ij} - r_i r_j\right)$



$\underline{r} = \left(\frac{a}{2}, \frac{a}{2}, 0\right)$

$$I_{ij} = \begin{pmatrix} \frac{5}{12}ma^2 & -\frac{ma^2}{4} & 0 \\ -\frac{ma^2}{4} & \frac{5}{12}ma^2 & 0 \\ 0 & 0 & \frac{2}{3}ma^2 \end{pmatrix}$$

Alternatively,

$$I_{zz} = I_{zz}^{SP} + m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{ma^2}{6} + \frac{ma^2}{2} = \frac{2}{3}ma^2$$

d) $L_{before} = m v_0 \frac{a}{2}$

$L_{after} = \frac{2}{3} ma^2 \omega_1$

$$L_{before} = L_{after}$$

$$\cancel{m} v_0 \frac{\cancel{a}}{2} = \frac{2}{3} \cancel{m} a^{\cancel{2}} \omega_1$$

$$\omega_1 = \frac{3}{4}\frac{v_0}{a}$$

e) $v_1 = \omega_1 \frac{a}{\sqrt{2}} = \frac{3}{4\sqrt{2}} v_0$

$E_K^{before} = \frac{m}{2} v_0^2$

$$E_K^{after} = \frac{1}{2} I_{zz}\, \omega_1^2 = \frac{1}{2}\,\frac{2}{3}\, m a^2 \left(\frac{3}{4}\,\frac{v_0}{a}\right)^2 = \frac{3}{16}\, m v_0^2$$

$$E_K^{before} - E_K^{after} = \frac{m}{2} v_0^2 - \frac{3}{16} m v_0^2 = \frac{5}{16} m v_0^2$$

f) $$\frac{1}{2} I_{zz}\, \omega_1^2 = mg\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right)$$

$$\frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\,\cancel{m}\cancel{a^2}\,\frac{9}{16}\,\frac{v_0^2}{\cancel{a^2}} = \cancel{m} g a\, \frac{\sqrt{2}-1}{2}$$

$$v_0^2 = \frac{8}{3}\left(\sqrt{2}-1\right) g a$$

$$v_0 = v_g = \sqrt{\frac{8(\sqrt{2}-1)}{3}\, g a}$$