Bosonization for Beginners

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In 1D, "bosonization relations" of the following form hold:

$$\underset{\text{fermion field}}{\psi} \sim \underset{\text{Klein factor}}{F}\; e^{-i\overset{\text{boson field}}{\phi}}$$

Goal of lectures:
- explain origin of these relations
- illustrate them with some canonical examples

Outline:
I. 1D-fermions, 1D-bosons
II. Bosonization identity
III. Impurity in Luttinger Liquid
IV. Kondo model

Literature:
- *Bosonization for Beginners - refermionization for experts*, Jan von Delft & Herbert Schoeller, Ann. Physics 7, 225-306 (1998), cond-mat/9805275
- *Simple Bosonization Solution of the 2-channel Kondo Model: I. Analytical Calculation of Finite-Size Crossover Spectrum*, Gergely Zarand and Jan von Delft, Phys. Rev. B. **61**, 6918 (2000) [including appendices: cond-mat/9812192]
- *Interacting fermions in one dimension: The Tomonaga-Luttinger model* K. Schönhammer, cond-mat/9710330

Popular applications

(pioneered by: Luttinger, Schotte & Schotte, Mattis &Lieb, Luther & Peschel, Haldane
applications: Kane & Fisher, Wen, Shankar...)

1. Interactions in 1D

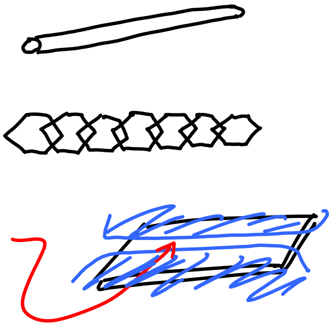
Since fermions in 1D cannot pass each other, interactions are "strong" and dramatically change the physics (e.g. spin-charge separation)

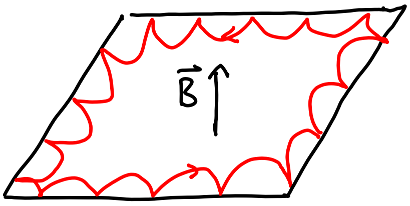
Applications:

nanotubes

organic molecules

semiconductor quantum wires





quantum Hall edge states

Interactions in 1D: $$\int dx\; \psi^\dagger \psi\; \overbrace{\psi^\dagger \psi}^{\partial_x \phi} \sim \int dx\, (\partial_x \phi)^2$$

Kinetic energy: $$\int dx\; \psi^\dagger \partial_x \psi \sim \int dx\, (\partial_x \phi)^2$$

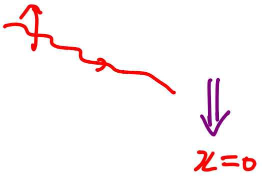
QUADRATIC !!

Interacting model becomes exactly solvable!

2. Impurity models (Kondo):

(Emery & Kivelson, '92)

$|\psi_{in}\rangle$:



$|\psi_{out}\rangle$:

Spin-flip term: $S^+ \psi^\dagger_\downarrow(0)\,\psi_\uparrow(0) \;+\; S^- \psi^\dagger_\uparrow(0)\,\psi_\downarrow(0)$

Bosonize: $\sim S^+ e^{i(\phi_\uparrow - \phi_\downarrow)} \;+\; S^- e^{\underbrace{i(\phi_\uparrow - \phi_\downarrow)}_{\phi_s}}$

$\psi_\sigma \sim e^{-i\phi_\sigma}$

New boson field: $\sim S^+ \underbrace{e^{-i\phi_s}}_{\psi_s(0)} \qquad S^- \underbrace{e^{i\phi_s}}_{\psi_s^\dagger(0)}$

Refermionize: $d^\dagger \psi_s \;+\; d\,\psi_s^\dagger$ QUADRATIC !!

Warning: I'm being sloppy here... See lecture 4

Heuristic plausibility argument for bosonization relation

How can it be true that: $\psi \sim e^{-i\phi}$? (1)

For 1-D bosons, with linear dispersion: $\langle \phi(x)\,\phi(0)\rangle \sim -\ln x$ (2)

For 1-D fermions, with linear dispersion: $\langle \psi(x)\,\psi^\dagger(0)\rangle \sim \frac{1}{x}$ (3)

$\big\downarrow (1)$

or, using (1): $\sim \langle e^{-i\phi(x)}\, e^{i\phi(0)}\rangle$ (4)

standard identity for bosonic operators: $\sim e^{\langle \phi(x)\phi(0) - \phi(0)\phi(0)\rangle}$ (5)

$\big\downarrow (2)$

using (2): $\sim e^{-\ln x} \sim \frac{1}{x} = (3)$ (6)

Questions:

$$\psi_\sigma \sim F_\sigma \, e^{-i\phi_\sigma} \qquad (\sigma = \uparrow, \downarrow) \quad (1)$$

How general is (5.1) ? only in 1D, infinite bandwidth

Does (5.1) rely on linear dispersion? NO!

Is (5.1) an operator identity? YES! On what Fock space ?

Commutation relations? $[\phi(x), \partial_x \phi(0)] = \delta(x) \Longleftrightarrow \{\psi(x), \psi^\dagger(0)\} = \delta(x)$

Several species of electrons? $\{\psi_\uparrow(x), \psi_\downarrow^\dagger(0)\} = 0 \Longleftrightarrow \{F_\downarrow, F_\uparrow^\dagger\} = 0$

Klein-factors

Role of cut-offs ? Infrared: $\frac{1}{L}$ Ultraviolet: $\Lambda \sim \frac{1}{a}$

Finite-size effects? $\frac{1}{L} \neq 0$ Useful

Outline of lecture I: 1-D fermions & bosons

1. Linearization of fermion spectrum
2. Properties of 1d fermion fields
4. Normal ordering
5. Density fluctuations - bosonic excitations

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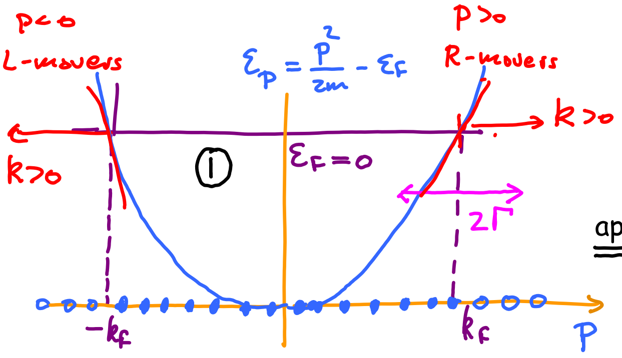
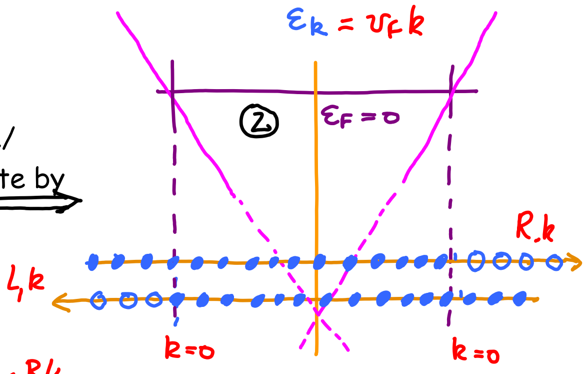
I.1 Linearization of fermion spectrum

(ignore spin)

$\Gamma = \infty$



replace/
approximate by



$k = |p| - k_F$, or $p = \pm(k_F + k)$ for R/L

add "positron states" with $k \to -\infty$ for L/R-branches

① For $|k| \ll \Gamma \simeq p_F$, linearization is justified:

$$\varepsilon_p = \frac{|p|^2 - p_F^2}{2m} = \frac{\cancel{p_F^2} + 2p_F k + k^2 - \cancel{p_F^2}}{2m} = v_F k\left(1 + \frac{k}{2p_F}\right) \quad (1)$$

curvature-effect small

$$\simeq v_F k \quad \text{if} \quad |k| < \Gamma \sim p_F \quad (2)$$

Neglected terms [order (k/kF)] describe curvature effects: current research topic!
Fermi-Luttinger liquid: Spectral function of interacting one-dimensional fermions, Khodas, Pustilnik, Kamenev, Glazman, PRB, 76, 155402 (2007)

Replacing (6.1) by (6.2) is justified if we are interested only in long-wavelength / low/energy properties, with $|k| \ll \Gamma$ anyway, i.e. in excitation energies $\omega, T, V \ll \varepsilon_F$

In this case, we may as well send cutoff $\Gamma \to \infty$ and replace theory ① → ②

Corresponding approximation for electron fields, step by step:

write $p = \pm(k_F+k)$

$$\psi_{phys}(x) = \Delta_L^{1/2}\sum_p e^{ipx} c_p = \Delta_L^{1/2}\sum_k \left(e^{-i(p_F+k)x} \underbrace{c_{-p_F-k}}_{c_{k,L}} + e^{i(p_F+k)x} \underbrace{c_{p_F+k}}_{c_{kR}} \right) \quad (1)$$

$\Delta_L = \left(\frac{2\pi}{L}\right)$

$$\hookrightarrow \simeq \left[\underbrace{\sum_{|k|<\Gamma}}_{Ⓐ} + \sum_{|k|>\Gamma} \cancel{\text{and } k > -k_F} \right]$$

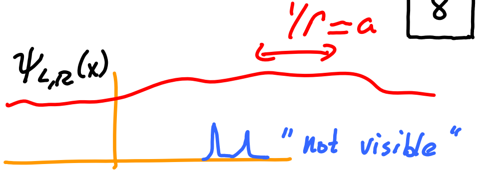
Drop high-energy excitations, assuming they don't matter for low-energy properties:

Step 1: drop B $\quad \to \quad \psi_{phys}(x) \text{ “=” } e^{-ip_Fx}\psi_L(x) + e^{ip_Fx}\psi_R(-x) \quad (2)$

with $\quad \psi_{L/R}(x) := \Delta_L^{1/2} \sum_{|k|<\Gamma} e^{-ikx} c_{k,L/R} \quad (3)$

I.2 Properties of 1d fermion fields

Cutoff means: new fields $\psi_{L/R}(x)$ can resolve spatial structures only if they are coarser than $1/\Gamma$;



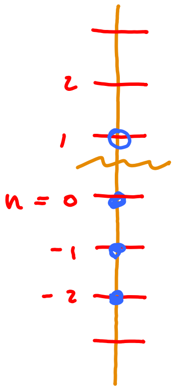
Step 2: to get a mathematically simpler, cleaner theory, now take cutoff to infinity, i.e. add "positron states" (since they did not matter for low excitation energies anyway):

$$\sum_{|k|<\Gamma=\frac{1}{a}} \longrightarrow \underset{\substack{\uparrow \\ \text{(implicit)}}}{\lim_{a\to 0}} \underbrace{\sum_{k=-\infty}^{\infty} e^{-|k|a}}_{\Sigma'}$$

So, write: $\eta = L, R$

(x is smeared on scale a)

$$\boxed{\psi_\eta(x) = \Delta^{1/2} \sum_k{}' e^{-ikx} c_{k\eta}} \quad (1)$$



Impose anti-periodic boundary conditions:
(convenient to avoid degeneracy of Fermi ground state)

$$\psi_\eta(-L/2) = -\psi_\eta(L/2) \Rightarrow k = \underbrace{\frac{2\pi}{L}}_{\Delta_L}(n - \tfrac{1}{2}) \quad (2)$$

Anticommutators: $\{c_{k\eta}, c_{k'\eta'}\} = 0, \quad \{c_{k\eta}, c^{\dagger}_{k'\eta'}\} = \delta_{kk'}\delta_{\eta\eta'}$ (1)

$$\{\psi_\eta(x), \psi_{\eta'}(x')\} = 0 \qquad (2)$$

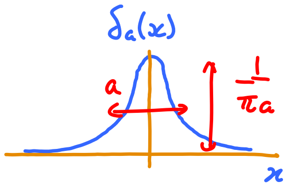
$$\{\psi_\eta(x), \psi^{\dagger}_{\eta'}(x')\} \overset{(8.1)}{=} \Delta_L \sum_{kk'}{}' e^{-ixk} e^{+ix'k'} \overbrace{\{c_{k\eta}, c^{\dagger}_{k'\eta'}\}}^{\delta_{kk'}\delta_{\eta\eta'}} \qquad (3)$$

$$= \delta_{\eta\eta'} \underbrace{\Delta_L \sum_k}_{\int_{-\infty}^{\infty} dk} e^{-ik(x-x')} e^{-|k|a} \qquad (4)$$

Continuum limit: (finite bandwidth) $\xrightarrow{L\to\infty}$

$$= \delta_{\eta\eta'} 2\pi \underbrace{\frac{a/\pi}{(x-x')^2 + a^2}}_{\delta_a(x-x')} \qquad (5)$$

convention of vDS $\quad = \delta_{\eta\eta'} 2\pi \; \delta_a(x-x')$



smeared delta-function

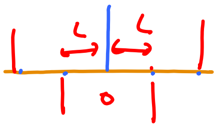
Or: $a \to 0$

infinite bandwidth $\{\psi_\eta(x), \psi^{\dagger}_{\eta'}(x')\} = \delta_{\eta\eta'} \Delta_L \sum_{n=\mathbb{Z}} e^{-i(n-1/2)(x-x')\Delta_L}$ (6)

$k = \Delta_L(n - 1/2)$

$n \in \mathbb{Z}$

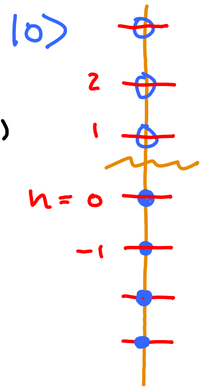
$$= \delta_{\eta\eta'} 2\pi \sum_{\bar{n}} \delta(x - x' - \bar{n}L)(-1)^{\bar{n}}$$



antiperiodic delta-function

Linearized kinetic energy: $H \overset{(6.2)}{=} \sum_{k\eta}{}' v_F k \, c^{\dagger}_{k\eta} c_{k\eta}$ (1)

Fermi ground state: $\begin{cases} k < 0 \text{ filled} : & c^{\dagger}_{k\eta}|0\rangle = 0 \\ k > 0 \text{ empty} : & c_{k\eta}|0\rangle = 0 \end{cases}$ (2)



Imaginary-time evolution: $c_{k\eta}(\tau) := e^{H\tau/\hbar} c_{k\eta} e^{-H\tau/\hbar}$ (3)

$$= e^{-\underbrace{(v_F/\hbar)}_{=1} k\tau} c_{k\eta} = e^{-k\tau} c_{k\eta} \qquad (4)$$

[If we ever need real-time evolution: $c_k(t) = c_k(\tau \to it)$ (5)]

Fermion field: $\psi_\eta(\tau, x) = \Delta_L^{1/2} \sum_k{}' e^{-k\underbrace{(ix+\tau)}_{:=z}} c_{k\eta} = \Delta_L^{1/2} \sum_k{}' e^{-kz} c_{k\eta} = \psi(z)$ (6)

I.3 Imaginary-time-ordered fermion correlator at T = 0

$$\psi_\eta(z) \overset{(10.6)}{=} \Delta_L^{1/2} \sum_k{}' e^{-kz} c_k$$

$$- \mathcal{G}_{\eta\eta'}(z) = \langle \mathcal{T} \psi_\eta(z) \psi^\dagger_{\eta'}(0) \rangle \quad (1)$$

$$= \theta(\tau) \langle \psi_\eta(z) \psi^\dagger_{\eta'}(0) \rangle - \theta(-\tau) \langle \psi^\dagger_{\eta'}(0) \psi_\eta(z) \rangle \quad (2)$$

$$= \Delta_L \sum_{kk'}{}' e^{-kz} \Big[\theta(\tau) \underbrace{\langle 0 | c_{k\eta} c^\dagger_{k'\eta'} | 0 \rangle}_{\delta_{kk'} \delta_{\eta\eta'} \theta(k')} - \theta(-\tau) \underbrace{\langle 0 | c^\dagger_{k'\eta'} c_{k\eta} | 0 \rangle}_{\delta_{kk'} \delta_{\eta\eta'} \theta(-k)} \Big] \quad (3)$$

$$= \delta_{\eta\eta'} \underbrace{\Delta_L \sum_{k>0}^{\infty}}_{\int_0^\infty dk} e^{-kz\sigma} \, \sigma \, e^{-ka} \qquad \sigma = \text{sign}(\tau) \quad (4)$$

$$\xrightarrow{L\to\infty} \delta_{\eta\eta'} \, \sigma \left[\frac{e^{-z\sigma - a}}{-z\sigma - a} \right]_0^\infty = \delta_{\eta\eta'} \frac{1}{z + \sigma a} \quad (5)$$

a regularizes
the correlator
for z = 0

For finite L one finds, using $k = \Delta_L (n + 1/2)$, $y := e^{-\Delta_L(\sigma z + a)}$:

$$- \mathcal{G}_{\eta\eta}(z) = \Delta_L \sigma \, y^{-\sigma/2} \sum_{n=0}^{\infty} y^n = \Delta_L \sigma \frac{y^{-(\sigma+1)/2}}{y^{-1/2} - y^{1/2}} = \frac{\delta_{\eta\eta'} e^{\pi(\sigma+1)/L}}{\frac{L}{\pi} \sinh\left[\frac{\pi}{L}(z + \sigma a)\right]} \quad (6)$$

I.4 Fermion normal ordering

$A, B, C \in \{ c_{k\eta}, c^\dagger_{k'\eta'} \}$

$$\begin{Bmatrix} c_{k\eta}, & \text{for} & k > 0 \\ c^\dagger_{k\eta}, & \text{for} & k < 0 \end{Bmatrix} \qquad \begin{Bmatrix} c_{k\eta}, & \text{for} & k < 0 \\ c_{k\eta}, & \text{for} & k > 0 \end{Bmatrix} \quad (1)$$

To bring " normal order" a product of operators,
move all operators that annihilate the vacuum to the right of all others,
and multiply by (-1) for each exchange of two fermion operators. (2)

For product of two operators,
this is equivalent to:

$$\overset{\times}{\underset{\times}{}} A B \overset{\times}{\underset{\times}{}} = AB - \langle 0 | A B | 0 \rangle \quad (3)$$

Example: $k > 0, k' < 0$:

$$\overset{\times}{\underset{\times}{}} c^\dagger_k c_{k'} \overset{\times}{\underset{\times}{}} = - c_{k'} c^\dagger_k \quad (4)$$

$$\overset{(9.1)}{=} c^\dagger_k c_{k'} - \underbrace{\delta_{kk'}}_{\langle c^\dagger_k c_{k'} \rangle} \overset{\checkmark}{=} (1) \quad (5)$$

By definition, vacuum
expectation value of two
normal ordered operators

$$\langle 0 | \overset{\times}{\underset{\times}{}} A B \overset{\times}{\underset{\times}{}} | 0 \rangle \overset{(3)}{=} 0 \quad (6)$$

I.4 Fermion normal ordering

$A, B, C \in \{c_{k\eta}, c^\dagger_{k'\eta'}\}$

too general!

To bring " normal order" a product of operators, subtract their vacuum expectation value:

$${}^{\times}_{\times} A B \cancel{C \ldots} {}^{\times}_{\times} = A B \cancel{C} \ldots - \langle 0 | A B \cancel{C \ldots} | 0 \rangle \qquad (1)$$

(see Fetter & Walecka)

$$\left\{ \begin{array}{ll} c_{k\eta}, & \text{for } k>0 \\ c^\dagger_{k\eta}, & \text{for } k<0 \end{array} \right\} \qquad \left\{ \begin{array}{ll} c_{k\eta}, & \text{for } k<0 \\ c^\dagger_{k\eta}, & \text{for } k>0 \end{array} \right\} \qquad (2)$$

Equivalently: move all operators that annihilate the vacuum to the right of all others, and multiply by (-1) for each exchange of two fermion operators. (3)

Example: $k>0, k'<0$:

$${}^{\times}_{\times} c^\dagger_k c_{k'} {}^{\times}_{\times} \overset{(3)}{=} - c_{k'} c^\dagger_k \qquad (4)$$

$$\overset{(9.1)}{=} c^\dagger_k c_{k'} - \underbrace{\delta_{kk'}}_{\langle c^\dagger_k c_{k'} \rangle} = (1) \qquad (5)$$

By definition, vacuum expectation value of

$$\langle 0 | {}^{\times}_{\times} A B {}^{\times}_{\times} | 0 \rangle \overset{(1)}{=} 0 \qquad (6)$$

I.5 Density fluctuations - bosonic excitations

(2 pi) density:

$$\rho_\eta(x) = {}^{\times}_{\times} \psi^\dagger_\eta(x) \psi_\eta(x) {}^{\times}_{\times} \qquad (1)$$

$$= \Delta_L \sum_{qk}{}' e^{i(\cancel{k}-q-\cancel{k})x} \overbrace{{}^{\times}_{\times} c^\dagger_{k-q\eta} c_{k\eta} {}^{\times}_{\times}}^{\text{bilenear, hence bosonic in character!}} \qquad (2)$$

Fourier representation: i.t.o. density modes:

$$= \overset{q=0}{\Delta_L \hat{N}_\eta} + \Delta_L \sum_{q>0} i \sqrt{n_q} \left(e^{-iqx} b_{q\eta} - e^{iqx} b^\dagger_{q\eta} \right) \qquad (3)$$

where we defined: $q = \Delta_L n_q \quad , \quad n_q \in \mathbb{N}_+$

Particle number relative to Fermi ground state:

$$\hat{N}_\eta = \sum_k {}^{\times}_{\times} c^\dagger_{k\eta} c_{k\eta} {}^{\times}_{\times} \qquad (4)$$

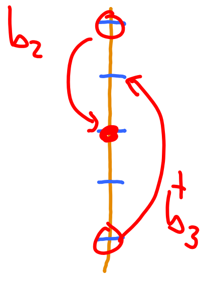
[the q = 0 term of (2)]

Momentum lowering op: (Bosonic annihilation op)

$$b_{q\eta} = \frac{-i}{\sqrt{n_q}} \sum_k c^\dagger_{k-q,\eta} c_{k,\eta} \qquad (5)$$

Momentum raising op: (Bosonic creation op)

$$b^\dagger_{q\eta} = \frac{i}{\sqrt{n_q}} \sum_k c^\dagger_{k+q,\eta} c_{k,\eta} \qquad (6)$$



Note: (5) and (6) are automatically normal ordered, hence no need to write ${}^{\times}_{\times} \quad {}^{\times}_{\times}$

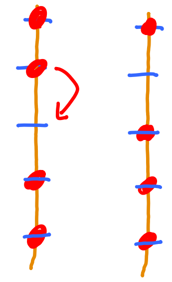
$$[A, BC]_- = [A,B]_\mp C \pm B[A,C]_\mp \qquad (1)$$

$$[a^\dagger b, c^\dagger d]_- = [a^\dagger b, c^\dagger]_- d + c^\dagger [a^\dagger b, d]_- = a^\dagger d \, \delta_{bc} - c^\dagger b \, \delta_{ad} \qquad (2)$$

Bosonic commutation relations: (for notational simplicity, below we drop the index η)

$q \neq 0$

$$[N, b_q] = \frac{-i}{\sqrt{n_q}} \sum_{kk'} [c^\dagger_k c_k, c^\dagger_{k'-q} c_{k'}] \qquad (3)$$



momentum-lowering operator does not change particle number

$$= \frac{-i}{\sqrt{n_q}} \sum_{kk'} \left(c^\dagger_k c_{k'} \delta_{k,k'-q} - c^\dagger_{k'-q} c_k \delta_{kk'} \right) \qquad (4)$$

cancel

$$= \frac{-i}{\sqrt{n_q}} \sum_{k} \left(c^\dagger_k c_{k+q} - c^\dagger_{k-q} c_k \right) = 0 \qquad (5)$$

shift: $k \to k+q$

Similarly:

$$[N, b^\dagger_q] = 0, \quad [b_q, b_{q'}] = 0, \quad [b^\dagger_q, b^\dagger_{q'}] = 0 \qquad (6)$$

$$\delta_{qq'} = [b_q, b^\dagger_{q'}] = \frac{1}{n_q} \sum_{kk'} [c^\dagger_{k-q} c_k, c^\dagger_{k'+q'} c_{k'}] \qquad (1)$$

$[a^\dagger b, c^\dagger d]_- = a^\dagger d \, \delta_{bc} - c^\dagger b \, \delta_{ad}$

$$= \frac{1}{n_q} \sum_{kk'} \left[c^\dagger_{k-q} c_{k'} \delta_{k,k'+q'} - c^\dagger_{k'+q'} c_k \delta_{k-q,k'} \right] \qquad (2)$$

$$= \frac{1}{n_q} \sum_{k} \left[c^\dagger_{k-q} c_{k-q'} - c^\dagger_{k-q+q'} c_k \right] \qquad (3)$$

cancel if $q \neq q'$

if $q \neq q'$, both terms are normal-ordered, so we can set $k+q' \Rightarrow k$ here, obtaining 0

if $q = q'$, both terms have to be normal-ordered first, before rearranging sum; this gives:

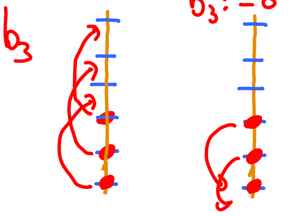
$$= \delta_{qq'} \frac{1}{n_q} \sum_k \left[{}^\times_\times c^\dagger_{k-q} c_{k-q} {}^\times_\times - {}^\times_\times c^\dagger_k c_k {}^\times_\times \Rightarrow 0 \right. \qquad (4)$$

number of possible transitions generated by

cancel after shift in first term: $k \to k+q$

$$\left. + \langle 0 | c^\dagger_{k-q} c_{k-q} | 0 \rangle - \langle 0 | c^\dagger_k c_k | 0 \rangle \right] \qquad (5)$$

$b^\dagger_3$ $\qquad b_3 := 0$



$k = \Delta_L (n_k - 1/2)$

$$= \delta_{qq'} \frac{1}{n_q} \left[\sum_{n_k = -\infty}^{n_q} - \sum_{n_k = -\infty}^{0} \right] = \frac{\delta_{qq'}}{n_q} \cdot n_q = \underline{\underline{\delta_{qq'}}} \qquad (6)$$

6. Properties of 1d Boson fields

$q = \Delta_L n_q,\ n_q \in \mathbb{N}_+$

"annihilation field": $$\varphi_\eta(x) = -\overbrace{\sum_{q>0} e^{-aq/2}}^{\sum'_q} \frac{1}{\sqrt{n_q}} e^{-iqx} b_{q\eta} \qquad (1)$$

"creation field": $$\varphi^\dagger_\eta(x) = -\sum_{q>0} e^{-aq/2} \frac{1}{\sqrt{n_q}} e^{+iqx} b^\dagger_{q\eta} \qquad (2)$$

The ultraviolet cutoff a here acts as a bandwidth for bosonic excitations.
In fermion language, it sets the maximum momentum difference between particle/hole pairs.

Hermitian boson field: $$\phi_\eta(x) = \varphi_\eta(x) + \varphi^\dagger_\eta(x) = \phi^\dagger_\eta(x) \qquad (3)$$

Derivative gives density:
(provided a = 0) $$\partial_x \phi_\eta(x) = \Delta_L \sum_q i\sqrt{n_q}\left(e^{-iqx} b_{q\eta} - e^{iqx} b^\dagger_{q\eta}\right) \qquad (4)$$

Compare (16.4) & (13.3): $$\rho_\eta(x) = {:}\varphi^\dagger_\eta(x)\,\varphi_\eta(x){:} = \Delta_L N_\eta + \partial_x \phi_\eta(x) \qquad (5)$$

Boson field commutators:

(for notational simplicity, below we drop the index η)

$$[b_q, b_{q'}] \overset{(14.6)}{=} [b^\dagger_q, b^\dagger_{q'}] = 0 \quad \Rightarrow \quad [\varphi(x), \varphi(x')] = 0 \ , \quad [\varphi^\dagger(x), \varphi^\dagger(x')] = 0 \ , \qquad (1)$$

$$[b_q, b^\dagger_{q'}] = \delta_{qq'} \quad \Rightarrow \quad [\varphi(x), \varphi^\dagger(x')] = \sum'_{qq'} \frac{1}{\sqrt{n_q}} \frac{1}{\sqrt{n_{q'}}} e^{-iqx} e^{iq'x'} \underbrace{[b_q, b^\dagger_{q'}]}_{\delta_{qq'}} \qquad (2)$$

$q = \Delta_L n$ $$= \sum_{q>0} \frac{1}{n_q} e^{-iq(x-x')} e^{-aq} \qquad (3)$$

$y = e^{-i\Delta_L(x - x' - ia)}$

$\xrightarrow{\Delta_L \to 0} 1 - i\Delta_L(\ \ )$ $$= \sum_{n=1}^{\infty} \frac{1}{n} y^n = -\ln(1-y) \qquad (4)$$

$\Delta_L = \frac{2\pi}{L}$ $$\xrightarrow[\Delta_L \to 0]{L\to\infty} -\ln\left(i\Delta_L(x - x' - ia)\right) \qquad (5)$$

$$[\varphi^\dagger(0), \varphi(0)] = \ln(\Delta_L a)$$

Note: this commutator needs both infrared and ultraviolet regulators, 1/L and a,

Commutator of phi with its derivative

$$\boxed{[\phi(x), \partial_{x'}\phi(x')]} = [\varphi(x), \partial_{x'}\varphi^\dagger(x')] + [\varphi^\dagger(x), \partial_{x'}\varphi(x')] \quad (1)$$

$$[\varphi+\varphi', \varphi+\varphi']$$

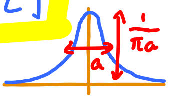
$$[\varphi(x), \varphi^\dagger(x)] \overset{(17.4)}{=} -\ln(1-y) = \left(i\Delta_L \frac{y}{1-y} - c.c\right) = i\Delta_L \frac{1}{y^{-1}-1} - c.c. \quad (2)$$

$$y = e^{-i\Delta_L(x-x'-ia)}$$

$$\xrightarrow{L\to\infty} i\cancel{\Delta}_L\left[\frac{1}{i\cancel{\Delta}_L(x-x'-ia)} - \frac{1}{2}\right] - c.c \quad (3)$$

(but retain 1/L term)

$$= i2\pi\left[\frac{a/\pi}{(x-x')^2+a^2} - \frac{1}{L}\right] = \boxed{2\pi i\left[\delta_a(x) - \frac{1}{L}\right]} \quad (4)$$



$$\frac{1}{1-e^\Delta} \qquad \Delta\to 0$$

$$= \frac{1}{1-(1+\Delta+\Delta^2/2+\dots)}$$

$$= \frac{1}{\Delta(1+\Delta/2)}$$

$$= \frac{1}{\Delta}(1-\Delta/2+\dots)$$

$$= \frac{1}{\Delta} - \frac{1}{2}$$

The 1/L term ensures consistency upon integrating (1):

$$\int_{-L/2}^{L/2} dx' \, [\phi(x), \partial_{x'}\phi(x')] \overset{(4)}{=} 2\pi i[1-1] = 0 \quad (19)$$

$$[\phi(x), \underbrace{\phi(L/2) - \phi(-L/2)}_{=0}] \quad (20)$$

consistent

since phi is periodic, (16.1, 16.2)

Commutator of phi with itself

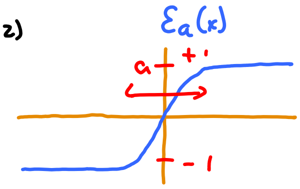
[can be obtained by integrating (18.1)]

$$[\phi(x), \phi(x')] = \int^{x'} d\bar{x} \, [\phi(x), \partial_{\bar{x}}\phi(\bar{x})] + c \quad (1)$$

fixed by requiring commutator to vanish for x = x'

$$\overset{(18.4)}{=} 2i\int^{x'} d\bar{x}\left[\frac{a}{(x-x')^2+a^2} - \frac{\pi}{L}\right] \quad (2)$$

$$= -2i\left[\arctan\left(\frac{x-x'}{a}\right) - \pi\frac{(x-x')}{L}\right] \quad (3)$$



smeared step function

$$= -i\pi\,\varepsilon_a(x-x') \quad (4)$$

where

$$\varepsilon_a(x) = \begin{cases} \pm 1 & \text{for } x \gtrless 0 \\ 0 & \text{for } x = 0 \end{cases} \quad (5)$$

(4) is the form most often quoted, with a = 0, L = infinity.