BCS 13

<u>Minimize</u>: $\partial_{\theta_k} E_{BCS} = 0$ (1)

since both sums $\sum_k$ and $\sum_{l'}$ make equal contributions

$$\xi_k (-2) \sin 2\theta_k + 2 \cdot \frac{2}{4} \sum V_{lk} \sin 2\theta_l \cos 2\theta_k \overset{(12.4)}{=} 0 \quad (2)$$

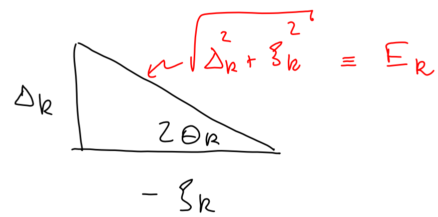
$\frac{(2)}{\cos 2k}$:

$$\tan 2\theta_k = \frac{1}{\xi_k} \underbrace{\sum_l V_{lk} \overbrace{\tfrac{1}{2}\sin 2\theta_l}^{(12.3):\ u_l v_l^*}}_{\equiv -\Delta_k} = -\frac{\Delta_k}{\xi_k} \quad (3)$$

where $\Delta_k \equiv -\sum_l V_{lk}\, u_l v_l$ (4)

Graphical representation of

$$\tan 2\theta_k \overset{(3)}{=} -\frac{\Delta_k}{\xi_k}:$$



(5)

BCS 14

(13.5) implies: $2 u_l v_l \overset{(12.3)}{=} \sin 2\theta_l \overset{(13.5)}{=} \frac{\Delta_l}{E_l}$ (1)

and for later use: $v_l^2 - u_l^2 \overset{(12.1)}{=} \cos^2\theta_l - \sin^2\theta_l = \cos 2\theta_l \overset{(14.1)}{=} -\frac{\xi_l}{E_l}$ (2)

[The choice of sign in (2) ensures $\left.\begin{matrix} u_l \to 1 \\ v_l \to 0 \end{matrix}\right\}$ for $\xi_l \to \infty$ as expected from page BCS 8 (3)

(14.1) into (13.4):

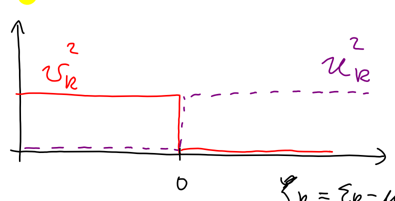
$$\boxed{\Delta_k = -\sum_l V_{kl} \frac{\Delta_l}{2E_l}}$$

"Gap equation" (4)

Trivial solution:

$\Delta_k = 0 \quad \forall k \quad \Rightarrow \quad E_k \overset{(13.5)}{=} |\xi_k|$

$v_k^2 - u_k^2 \overset{(14.2)}{=} -\frac{\xi_k}{|\xi_k|} = -\mathrm{sgn}(\xi_k)$



$\Rightarrow \quad v_k = \theta(-\xi_k), \quad u_k = \theta(\xi_k) \quad \Rightarrow$ FREE FERMI SEA.

Nontrivial solution has $\Delta_k \neq 0$ (to be determined below). $\boxed{BCS15}$

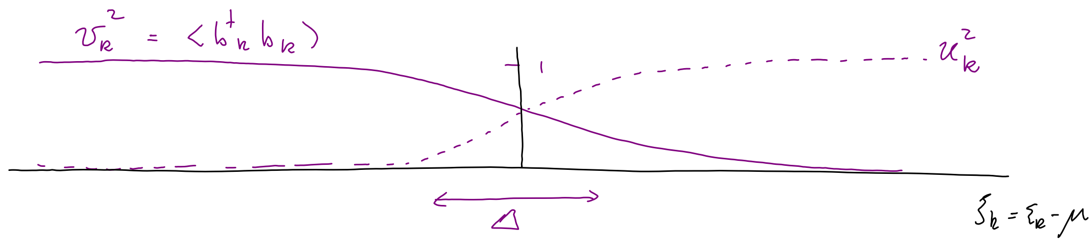
Find u_k, v_k:

$$v_k^2 + u_k^2 \overset{(8.2)}{=} 1 \qquad (1)$$

$$v_k^2 - u_k^2 \overset{(14.2)}{=} -\frac{\xi_k}{E_k} \qquad (2)$$

$\frac{1}{2}\left[(1) \pm (2)\right]$:

$$\left.\begin{matrix} v_k^2 \\ u_k^2 \end{matrix}\right\} = \frac{1}{2}\left(1 \mp \frac{\xi_k}{E_k}\right) = \frac{1}{2}\left(1 \mp \frac{\xi_k}{\sqrt{\Delta^2 + \xi_k^2}}\right)$$



Pair correlations "reshuffle" electrons from below to above the Fermi level.

This costs kinetic energy but gains potential energy!

To evaluate non-trivial solution with $\Delta_k \neq 0$, use BCS-model: $\boxed{BCS15}$

$$V_{kk} \overset{(5.2)}{=} \begin{cases} -V_0 \; (<0) & \text{for } |\xi_k| < \Omega \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

$$\Longrightarrow \quad \Delta_k \overset{(14.4)}{=} \begin{cases} \Delta & \text{for } |\xi_k| < \Omega \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

$\xi = \varepsilon - \mu$

with
$$1 \overset{(14.4)}{=} V_0 \sum_{|\xi_k| < \Omega} \frac{1}{2E_k} \qquad (3)$$

Now: $\left(\delta k = \frac{2\pi}{L}\right)$

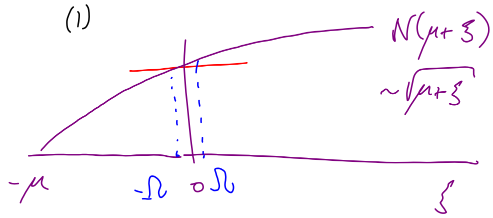
$$\sum_k = \frac{1}{(\delta k)^3}\int d^3k = \frac{L^3}{(2\pi)^3}\int_0^\infty dk\, k^2 \cdot 4\pi = \int_0^\infty d\varepsilon_k\, \mathcal{N}(\varepsilon_k) = \int_{-\mu}^\infty d\xi\, \mathcal{N}(\mu+\xi) \qquad (4)$$

where
$$\varepsilon_k = \frac{k^2}{2m} \Longrightarrow \frac{d\varepsilon_k}{dk} = \frac{k}{m} \Longrightarrow \sqrt{2m\varepsilon_k}\, d\varepsilon_k = dk\, \frac{k}{m}\cdot k \qquad (5)$$

$$\Longrightarrow \frac{L^3}{4\pi^2}\, dk\, k^2 = \frac{L^3}{4\pi^2}\sqrt{2m^3\,\varepsilon_k}\, d\varepsilon_k \equiv \mathcal{N}(\varepsilon_k)\, d\varepsilon_k = \mathcal{N}(\mu+\xi)\, d\xi \qquad (6)$$

Hence: [BCS17]

$$1 \overset{(15.3)}{=} V_0 \frac{1}{2} \int_{-\mu}^{\infty} d\xi \, \mathcal{N}(\mu+\xi) \frac{\theta(\Omega - |\xi|)}{\sqrt{\Delta^2 + \xi^2}} \qquad (1)$$



$$\simeq V_0 \cancel{\frac{1}{2}} \int_{\cancel{-\Omega}\; 0}^{\Omega} d\xi \, \mathcal{N}(\mu) \frac{1}{\sqrt{\Delta^2 + \xi^2}} \qquad (2)$$

assume: $\Delta \ll \Omega \ll \mu$

$$\frac{1}{V_0 \mathcal{N}(\mu)} = \sinh^{-1} \underbrace{\frac{\sqrt{\Delta^2 + \Omega^2}}{\Delta}}_{} \Big\} \simeq \Omega \qquad (3)$$

$$\left[\sinh\frac{1}{x} = \frac{1}{2}\left(e^{\frac{1}{x}} - e^{-\frac{1}{x}}\right) \simeq \frac{1}{2} e^{\frac{1}{x}} \quad \text{for } x \ll 1 \right]$$

$$\Delta = \Omega \left[\sinh\left(\frac{1}{V_0 \mathcal{N}(\mu)}\right)\right]^{-1} = 2\Omega \, e^{-\frac{1}{V_0 \mathcal{N}(\mu)}} \qquad (4)$$

(since typically, $V_0 \mathcal{N}(\mu) \lesssim 0.3$, approx OK with error of $\lesssim 1\%$)

We get a non-perturbative result for Δ!!

Condensation energy: [BCS18]

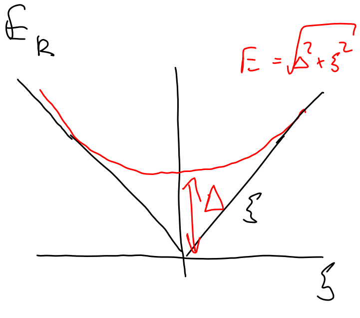
$$\Delta E_{BCS} \overset{(12.4)}{=} \langle BCS | \hat{H} - \mu \hat{N} | BCS \rangle - \langle F | \hat{H} - \mu \hat{N} | F \rangle \qquad (1)$$

$$= \left[\sum_k \left(\xi_k - \frac{\xi_k^2}{E_k} \right) - \frac{\Delta^2}{V_0} \right] - \sum_{|\varepsilon_k| < \varepsilon_F} 2 \xi_k$$

Exercise:

$$= -\frac{1}{2} \Delta^2 \mathcal{N}(\mu) \quad \sim \text{Volume}, \quad \text{hence extensive gain!}$$

Excited states: $c^\dagger_{k\sigma} |BCS\rangle$, $c_{k\sigma} |BCS\rangle$ } both have energy E_k

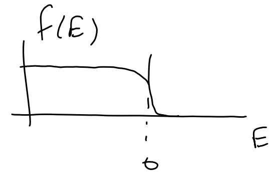


Finite temperature gap equation:

(obtained by minimizing free energy $H - TS$):

$$1 = -\sum_{l} V_{kl} \frac{\Delta_l}{2E_l}\left(1 - 2f(E_l)\right) \qquad (1)$$

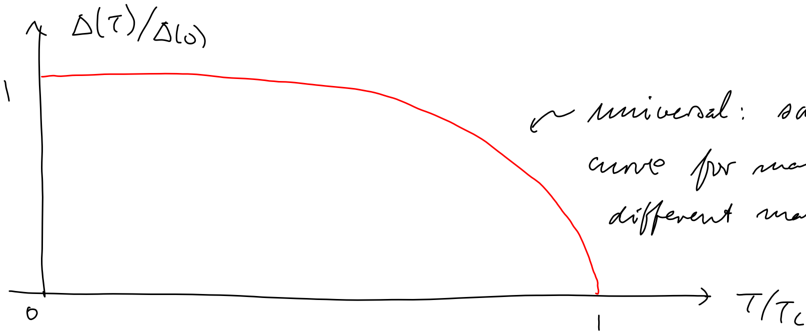
where $f(E_l) = \dfrac{1}{e^{\beta E_l} + 1}$ = Fermi function



(2)

Solution:

(can be shown)



universal: same curve for many different materials!