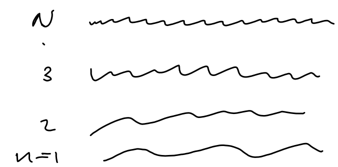
Dissipative Quantum Mechanics – Spin-Boson model 16.11.09 Dis 1

Literature:

U. Weiss: Quantum Dissipative Systems, 2nd Edition (1999) (World Scientific, Singapore)

A. Leggett: Dynamics of the Dissipative Two-State System, Rev. Mod. Phys., 59, 1 (1987)

Finite Quantum Systems are always coherent: energy is conserved, and is transferred back and forth between degrees of freedom:



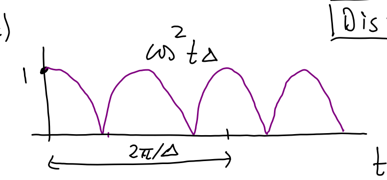
$$|\psi(t)\rangle = \sum_{n=1}^{N} e^{-iE_n t} |n\rangle \qquad (\hbar = 1)$$

$$|\psi(t)\rangle = |\psi(0)\rangle \quad \text{if} \quad E_n t = 2\pi n' \quad \forall\, n$$

Return time is finite: approximate E_n's by rational numbers: $E_n \simeq \frac{K_n}{M_n}$

$\Rightarrow$ for $t = 2\pi \prod_{n=1}^{N} M_n$, we have $E_n t = 2\pi K_n \prod_{n' \neq n}^{N} M_{n'}$ = of form $2\pi n'$.

Simple Example: 2-State System: Dis 2



$$H = -\frac{1}{2}\begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} = -\frac{1}{2}\sigma_x \tag{1}$$

Eigenstates: $|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (2)

Eigenenergies: $E_1 = -\frac{1}{2}\Delta$, $E_2 = \frac{1}{2}\Delta$ Splitting: $E_2 - E_1 = \Delta$ (3)

Suppose $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$ (4)

Time evolution:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_1 t}|1\rangle + e^{-iE_2 t}|2\rangle\right) \tag{5}$$

$$P_{\binom{1}{0}}(t) = |\langle\psi(0)|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(e^{-iE_1 t}\frac{1}{\sqrt{2}} + e^{-iE_2 t}\frac{1}{\sqrt{2}}\right)\right|^2 \tag{6}$$

$$= \left|e^{-i(E_1+E_2)t/2}\,\frac{1}{2}\left(e^{-it(E_1-E_2)/2} + e^{it(E_1-E_2)/2}\right)\right|^2 = \cos^2 t\Delta \tag{7}$$

System oscillates forever between its two eigenstates. (8)

Depiction on Bloch sphere

Dis3

Write $H = -\vec{S}\cdot\vec{B}$ $\qquad \left(\vec{S} = \frac{1}{2}\vec{\sigma};\ B_x = \Delta,\ B_y = 0,\ B_z = 0\right)$ (1)

Heisenberg eq. of motion: $\partial_t \underbrace{S_\ell}_{\ell = x,y,z}(t) = -i\,[S_\ell, H] = -i\,(-)\sum_m [\underbrace{S_\ell, S_m}_{} B_m]$ (2)

$i\varepsilon_{\ell m n} S_n B_m = i(\vec{B}\times\vec{S})_\ell$

$$\boxed{\partial_t \vec{S} = \vec{S}\times\vec{B}} \qquad (3)$$

$$\dot{S}_x = S_y B_z - S_z B_y = 0 \qquad (4)$$

$$\dot{S}_y = S_z B_x - S_x B_z = \Delta S_z \qquad (5)$$

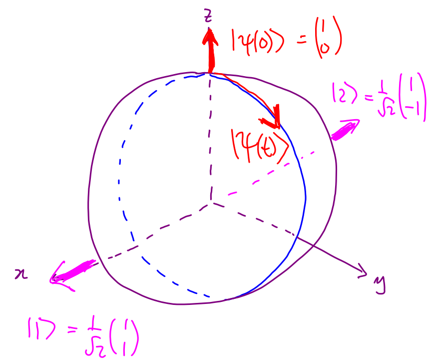
$$\dot{S}_z = S_x B_y - S_y B_x = -\Delta S_y \qquad (6)$$

$$\ddot{S}_z = -\Delta \dot{S}_y = \Delta^2 S_z \qquad (7)$$

$$S_z(t) = \cos\Delta t\; S_z(0) \qquad (8)$$

$\Rightarrow$

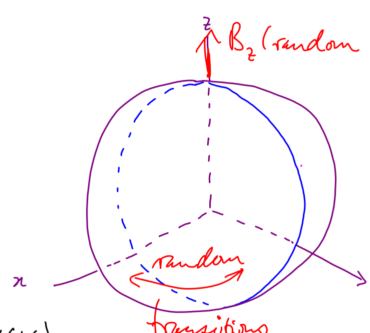
$$P(t) \equiv \langle \hat{z}\cdot\vec{S}(t)\rangle = \langle S_z(t)\rangle = \cos\Delta t\,\langle S_z(0)\rangle$$



Dissipation: Bloch Eqs:

Dis4

We would like to describe damping.
Suppose there is a random fluctuating field in z-direction; it will rotate $S_z \leftrightarrow S_y$ into each other, causing both to decay.
Phenomenological description: "Bloch equations" (1946)



$$\dot{S}_x = -\frac{1}{T_1}\left(S_x - S_x^{eq}\right) \qquad (1)$$

T_1 : rate for energy-changing transitions
S_x^{eq} : equilibrium value of S_z.

$$\dot{S}_y = \Delta S_z - \frac{1}{T_2} S_x \qquad (2)$$

T_2 : rate for energy-conserving transit.

$$\dot{S}_z = -\Delta S_y \qquad (3)$$

(no damping, since random field is in z-direction)

Solution of Bloch eqs:

Dis 5

(4.1) : exp. decay :
$$S_x(t) = S_x^{eq} + \left[S_x(0) - S_x^{eq}\right] e^{-t/T_1}$$

$-\frac{1}{\Delta}\,\partial_t(4.3)$:
$$\ddot{S}_z \overset{\partial_t(4.3)}{=} -\Delta \dot{S}_y \overset{(4.2)}{=} -\Delta\Big(\Delta S_z - \frac{1}{T_2}\underbrace{S_y}_{(4.3)\ -\dot{S}_z/\Delta}\Big) \qquad (1)$$

$$\ddot{S}_z + \frac{\dot{S}_z}{T_2} + \Delta^2 S_z = 0 \qquad (2)$$

$S_z(t)$ performs damped harmonic oscillations with damping constant $\gamma = \frac{1}{T_2}$.

Applett: http://comp.uark.edu/~jgeabana/blochapps/blocheqs2.html

To make contact with our notation, take: $\mathrm{Real}(\Omega) = \Delta_{\text{here}}$, $\mathrm{Im}(\Omega) = 0$,

$\gamma_1 = 0$, $\gamma_2 = \frac{1}{T_2} = \frac{1}{T_1}$, $\Delta_{\text{applett}} = 0$

Dis6

To describe dissipation, quantum mechanically, we need a "bath" with infinitely many degrees of freedom. Then "system" can loose energy to bath, and "return time" becomes ∞.

Experience shows: one useful way to model dissipation is to couple system to a bath of harmonic oscillators:

Famous Example: Spin-boson model : (Leggett, Rev. Mod. Phys., 1987)

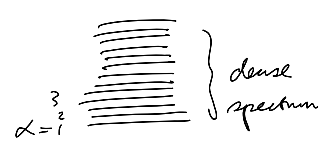
$$\underset{\uparrow\ \text{spin, system}}{H_S} = -\tfrac{1}{2}\Delta\sigma_x + \tfrac{1}{2}\varepsilon\sigma_z \qquad (1)$$
(we will take $\varepsilon = 0$ for simplicity)

$$H_{bath} = \sum_\alpha \omega_\alpha \left(b_\alpha^\dagger b_\alpha + 1/2\right) \qquad (2)$$

$$H_{int} = \underbrace{\tfrac{q}{2}\sigma_z}_{S_z}\ \underbrace{\sum_\alpha \lambda_\alpha \underbrace{(b_\alpha^\dagger + b_\alpha)}_{\sim\, x_\alpha}}_{\tilde{B}_z} \qquad (3)$$

$\sim x_\alpha$: oscillators "shake" spin.

like a fluctuating B_z-field (4)



$\omega_{\alpha+1} - \omega_\alpha = \delta\omega \to 0$

Golden rule calculation of decay rates T_1, T_2

Dis 7

In (4.1), $1/T_1$ is the rate for flipping $S_x \to \leftarrow S_x$, or $|1\rangle \Leftrightarrow |2\rangle$ (1)

(4.2) $1/T_2$ $S_y \leftrightarrow -S_y$ (2)

[for spin-boson model, we expect $T_1 = T_2$, since $\sigma_z \tilde{B}_z$ produces rotations around z-Axis]

Golden rule calculation for T_1:

$$\frac{1}{T_1} = 2\pi \sum_{fi} P(i) \left| \langle f | H_{int} | i \rangle \right|^2 \delta(\varepsilon_f - \varepsilon_i) \qquad (3)$$

↑ probability to be in initial state $|i\rangle$

Form of $\left\{ \begin{matrix} \text{initial} \\ \text{final} \end{matrix} \right\}$ states:

$$|i\rangle = |\sigma\rangle \otimes |B\rangle, \qquad |f\rangle = |\bar{\sigma}\rangle \otimes |B'\rangle \qquad (4)$$

$\left\{ \begin{matrix} |\sigma\rangle = |1\rangle \text{ or } |2\rangle \\ |\bar{\sigma}\rangle = |2\rangle \text{ or } |1\rangle \end{matrix} \right.$

arbitrary bath states

$$|B\rangle = \prod_{\{n_1, n_2 \dots\}} |n_1\rangle |n_2\rangle \dots$$

$$|B\rangle = \prod_{\{n_1, n_2, \dots\}} |n_1\rangle |n_2\rangle \dots$$

Dis 8

$$\frac{1}{T_1} = 2\pi \sum_{BB'} P(B) \sum_{\sigma=1,2} \Big| \underbrace{\langle \bar{\sigma} | \langle B' | H_{int} | \sigma \rangle | B \rangle}_{} \Big|^2 \delta(E_{\bar{\sigma}} + E_{B'} - E_\sigma - E_B) \qquad (1)$$

$$\to q \underbrace{\langle \bar{\sigma} | \tfrac{1}{2} \sigma_z | \sigma \rangle}_{= 1/2} \langle B' | \sum_\alpha \lambda_\alpha (b_\alpha^\dagger + b_\alpha) | B \rangle \qquad (2)$$

$$= 2\pi \frac{q^2}{4} \sum_{BB'} P(B) \sum_{\alpha\alpha'} \lambda_{\alpha'}^* \lambda_\alpha \langle B | b_{\alpha'}^\dagger + b_{\alpha'} | B' \rangle \langle B' | b_\alpha^\dagger + b_\alpha | B \rangle \sum_\sigma \delta(E_{\bar{\sigma}} - E_\sigma + E_{B'} - E_B) \qquad (3)$$

$$= \pi q/2 \sum_{BB'} P(B) \sum_{\alpha\alpha'} |\lambda_\alpha|^2 \delta_{\alpha\alpha'} \Big[\langle B | b_{\alpha'}^\dagger | B' \rangle \langle B' | b_\alpha | B \rangle \, \delta(\overbrace{E_2 - E_1}^{\Delta} - \omega_\alpha) + \langle B | b_{\alpha'} | B' \rangle \langle B' | b_\alpha^\dagger | B \rangle \, \delta(\overbrace{E_1 - E_2}^{-\Delta} + \omega_\alpha) \Big. \qquad (4)$$

$\left[\begin{matrix} E_{B'} = E_B - \omega_\alpha, & \sigma = 1, \\ E_{B'} = E_B + \omega_\alpha, & \sigma = 2 \end{matrix} \right]$

$|B\rangle$ and $|B'\rangle$ can differ only by one boson, hence $\alpha = \alpha'$

$$= q^2 \frac{\pi}{2} \sum_\alpha |\lambda_\alpha|^2 \underbrace{\sum_B P(B) \Big[\langle B | b_\alpha^\dagger b_\alpha | B \rangle}_{\bar{n}_\alpha = [e^{\omega_\alpha/T} - 1]^{-1} = \text{average boson number in mode } \alpha} + \underbrace{\langle B | b_\alpha b_\alpha^\dagger | B \rangle}_{b_\alpha^\dagger b_\alpha + 1 \to \bar{n}_\alpha + 1} \Big] \delta(\omega_\alpha - \Delta) \qquad (5)$$

$$\sim (2\bar{n}_\alpha + 1) = \frac{2 + e^x - 1}{e^x - 1} = \coth \frac{\omega_\alpha}{2T}$$

$$= q^2 \frac{\pi}{2} \sum_\alpha |\lambda_\alpha|^2 \delta(\omega_\alpha - \Delta) \coth \frac{\omega_\alpha}{2T} \qquad (6)$$

$$= \frac{q^2}{2} \coth\left(\frac{\Delta}{2T}\right) \left[\pi \sum_\alpha |\lambda_\alpha|^2 \delta(\omega_\alpha - \Delta) \right] \qquad (7)$$

$\equiv J(\Delta)$ = "spectral function" = characterizes strength of bath-spin coupling

Final result:

$$\boxed{1/T_1 = q^2 \tfrac{1}{2} \coth(\Delta/2T) \, J(\Delta)} \qquad (8)$$