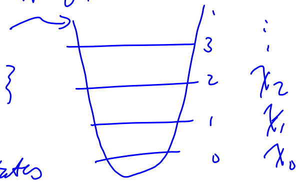
Bose-Einstein condensates (BEC) : some basic properties

13.12.09

[Leggett, RMP 2001]

Ground state for a BEC with N noninteracting particles.

e.g. trapping potential



$\{\chi_i(\vec r),\ i=0,\dots\infty\}$ is complete set of 1-particle eigenstates

$$\boxed{|G\rangle_N = (a_0^\dagger)^N |Vac\rangle} \quad (1)$$

Wave-function:

$$\boxed{\Psi_g(\vec r_1,\dots,\vec r_N;t) = \prod_{j=1}^{N} \chi_0(\vec r_j;t)} \quad \text{(automatically symmetric)} \quad (2)$$

[must be symmetric under $\vec r_i \leftrightarrow \vec r_j$]

all particles are in the same single-particle state !

$$\int d\vec r_1 \dots \int d\vec r_N\ \Psi_g^*(\vec r_1,\dots \vec r_N;t)\,\Psi_g(\vec r_1,\dots \vec r_N;t) = \prod_{j=1}^{N} \underbrace{\int d\vec r_j\ \chi_0^*(\vec r_j;t)\,\chi_0(\vec r_j;t)}_{=1} = 1 \quad (3)$$

by normalization of single-particle wave-function

Density operator: (1st-quantized form)

$$\hat\rho(\vec r) = \sum_{j=1}^{N} \delta(\vec r - \vec r_j) \quad (1)$$

density :

$$\rho^{(1)}(\vec r;t) = \int d\vec r_1 \dots \int d\vec r_N\ \hat\rho(\vec r)\, |\Psi_g(\vec r_1,\dots \vec r_N)|^2 \quad (2)$$

$$= \sum_{j=1}^{N} |\chi_0(\vec r,t)|^2 = N\,|\chi_0(\vec r,t)|^2 \quad (3)$$

Total particle number:

$$N_{tot} = \int d\vec r\ \rho^{(1)}(\vec r;t) = N \underbrace{\int d^3\vec r\ \chi_0^*(\vec r;t)\,\chi_0(\vec r;t)}_{=1} = 1 \quad (4)$$

Single-particle density matrix:

$$\rho(\vec r,\vec r';t) \equiv N \int d\vec r_2 \dots \int d\vec r_N\ \psi_g^*(\vec r,\vec r_2,\dots \vec r_N)\ \psi_g(\vec r',\vec r_2,\dots,\vec r_N) \quad (5)$$

$$\chi_0^*(\vec r')\prod_{j=2}^{N} \chi_j^*(\vec r_j)\ \chi_0(\vec r')\prod_{j=2}^{N} \chi_j(\vec r_j)$$

$$= N\,\chi_0^*(\vec r,t)\,\chi_0(\vec r',t) \quad (6)$$

"off-diagonal long range order"

($\neq 0$ for $\vec r - \vec r'$)

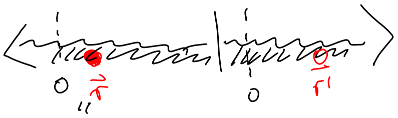
Define
$$\hat\psi(\vec r,t) = \sum_{j=0}^{\infty} \chi_j(\vec r,t)a_j \qquad (1)$$
$$\hat\psi^\dagger(\vec r,t) = \sum_{j=0}^{\infty} \chi_j^*(\vec r,t)a_j^\dagger \qquad (2)$$

"field operators: destroys/create particle "at" $\vec r$ "

Density matrix can also be defined as (this is equivalent to 2.2):

$$\rho(\vec r,\vec r\,';t) \equiv \langle g|\,\hat\psi^\dagger(\vec r,t)\,\hat\psi(\vec r\,',t)|g\rangle \qquad (3)$$
$$= \sum_{jj'} \chi_j^*(\vec r,t)\chi_{j'}(\vec r\,',t)\,\underbrace{\langle g|\,a_j^\dagger a_{j'}|g\rangle}_{N\delta_{j0}\delta_{j'0}} \qquad (4)$$
$$= N\,\chi_0^*(\vec r,t)\chi(\vec r\,';t) \qquad (5)$$

$\rho(\vec r,\vec r\,',t)\neq 0$ for $\vec r \neq \vec r\,'$ means:



$\neq 0$

"off-diagonal long range order"

"Order parameter":

$$\boxed{\Psi(\vec r,t) \equiv \sqrt{N_0}\,\chi_0(\vec r,t) = \sqrt{N_0}\,|\chi_0(\vec r,t)|\,e^{i\overset{\text{phase}}{\overset{\downarrow}{\varphi_0(\vec r,t)}}}} \qquad (1)$$

$N_0 = \langle g|\,a_0^\dagger a_0\,|g\rangle$ = # of particles in single-particle g.s. (2)

Note: $\Psi \sim \chi_0 \simeq$ wave-function! So order parameter "is" a wave function.

Velocity:

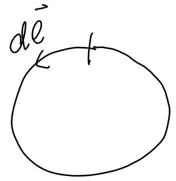
$$v(\vec r) = \int d\vec r_2\ldots\int d\vec r_N\;\Psi_g^*(\vec r,\vec r_2\ldots,\vec r_N)\left(\frac{-i\hbar\vec\nabla_{\vec r}}{m}\right)\Psi_g(\vec r,\vec r_2\ldots,\vec r_N) \qquad (3)$$
$$= \chi_0(\vec r)\left(\frac{-i\hbar\vec\nabla_{\vec r}}{m}\right)\chi_0(\vec r) = \vec v_N + \vec v_S \qquad (4)$$

"normal" velocity: $\vec v_N(\vec r)\qquad |\chi_0(\vec r)|\vec\nabla|\chi_0(\vec r)| = \frac12\vec\nabla\,|\chi_0(\vec r)|^2$ (5)

"superfluid" velocity: $\vec v_S = \frac{\hbar}{m}\vec\nabla\varphi(\vec r)$ (6)

So even if density is completely uniform, so $\vec{v}_N(\vec{r}) = 0$,
we can have <u>flow</u> if <u>phase</u> of order parameter changes with $\vec{r}$.

Properties of $\vec{v}_s$: $\vec{\nabla} \times \vec{v}_s = \frac{\hbar}{m} \vec{\nabla} \times (\vec{\nabla} \varphi) = 0$ "irrotational flow"



$$\oint d\vec{l} \cdot \vec{v}_s = \frac{\hbar}{m} \oint d\vec{l} \cdot \frac{d\varphi}{d\vec{l}} = \frac{\hbar}{m} \delta\varphi$$

$\delta\varphi$ is change of phase of wave-function after going around loop.

Since $\chi_0(\vec{r})$ is single-valued, we need $\chi_0(\vec{r}) = \chi_0(\vec{r}) e^{i\delta\varphi(\vec{r})}$

$\Rightarrow \delta\varphi = 2\pi n$ (n integral)

$$\boxed{\oint d\vec{l} \cdot \vec{v}_s = \frac{h}{m} n}$$

Onsager-Feynman quantization condition

Observation of Vortex Lattices in Bose-Einstein Condensates

J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle

Quantized vortices play a key role in superfluidity and superconductivity. We have observed the formation of highly ordered vortex lattices in a rotating Bose-condensed gas. These triangular lattices contained over 100 vortices with lifetimes of several seconds. Individual vortices persisted up to 40 seconds. The lattices could be generated over a wide range of rotation frequencies and trap geometries, shedding light on the formation process. Our observation of dislocations, irregular structure, and dynamics indicates that gaseous Bose-Einstein condensates may be a model system for the study of vortex matter.

Science, 2001

- "stir" a BEC
- quantization of rotational flow
  $\Rightarrow$ vortices !!
- fluid attempts to distribute vorticity as uniformly as possible

$\Rightarrow$ (triangular) vortex lattice forms !!

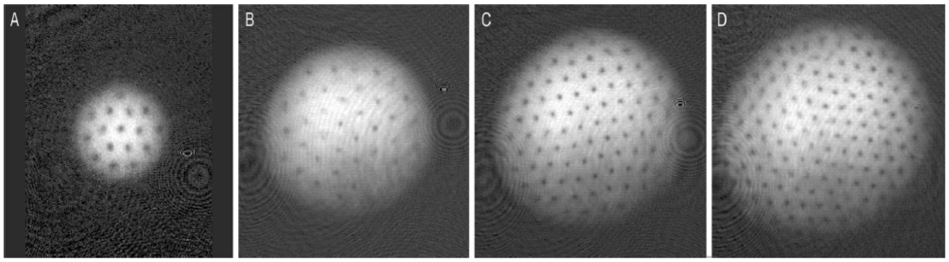


Fig. 1. Observation of vortex lattices. The examples shown contain approximately (**A**) 16, (**B**) 32, (**C**) 80, and (**D**) 130 vortices. The vortices have "crystallized" in a triangular pattern. The diameter of the cloud in (D) was 1 mm after ballistic expansion, which represents a magnification of 20. Slight asymmetries in the density distribution were due to absorption of the optical pumping light.

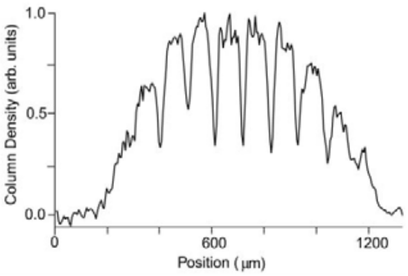
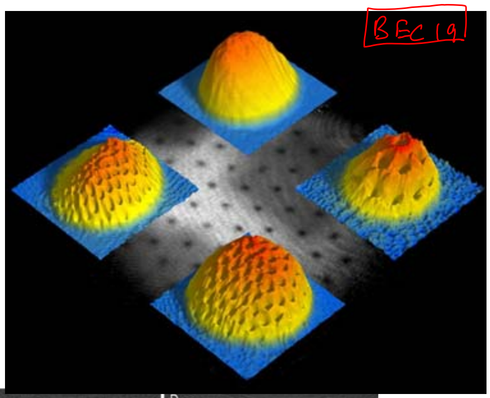


Fig. 2. Density profile through a vortex lattice. The curve represents a 5-μm-wide cut through a two-dimensional image similar to those in Fig. 1 and shows the high contrast in the observation of the vortex cores. The peak absorption in this image is 90%.



Time-evolution of vortex lattice

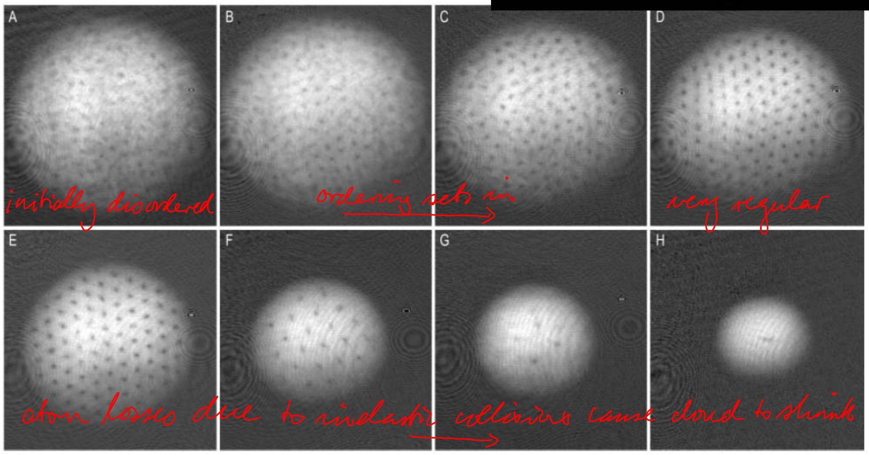


Fig. 4. Formation and decay of a vortex lattice. The condensate was rotated for 400 ms and then equilibrated in the stationary magnetic trap for various hold times. (**A**) 25 ms, (**B**) 100 ms, (**C**) 200 ms, (**D**) 500 ms, (**E**) 1 s, (**F**) 5 s, (**G**) 10 s, and (**H**) 40 s. The decreasing size of the cloud in (E) to (H) reflects a decrease in atom number due to inelastic collisions. The field of view is ~1 mm by 1.15 mm.

Interactions: (assume point-like interaction)

$$\hat{H} = \sum_{j=1}^{N}\left[-\frac{\hbar^2}{2m}\vec{\nabla}_j^2 + V_{ext}(\vec{r}_j)\right] + \sum_{i=1}^{N}\sum_{j<i}\delta(\vec{r}_i-\vec{r}_j)\,\underset{\uparrow}{U} \qquad (1)$$

repulsive contact interaction

(1st.-quantized form)

Variational "Hartree-Fock-Ansatz" for ground state wave function:

$$\Psi_g(\vec{r}_1,\dots\vec{r}_N) = \prod_{j=1}^{N}\chi_0(\vec{r}_j) \qquad (2)$$

[as in (13.2), but now χ_0 is not known a priori; to be found variationally!]

[with $\int d\vec{r}\,|\chi_0(\vec{r})|^2 = 1$]

$$\langle\hat{H}\rangle_N = \int d\vec{r}\left\{\frac{N\hbar^2}{2m}\chi_0^*(\vec{r})\left(-\vec{\nabla}^2\chi_0\right) + N V_{ext}(\vec{r})\,|\chi_0(\vec{r})|^2 + \overbrace{\tfrac{1}{2}N(N-1)}^{\#\text{ of pairs}}\,\overbrace{|\chi_0(\vec{r})|^4}^{\text{prob. for two atoms to be at }\vec{r}}\right\} \qquad (3)$$

Minimize with constraint of fixed average particle number

$$\langle N\rangle = N\int d\vec{r}\,|\chi_0(\vec{r})|^2,$$ using Lagrange multiplier μ

$$\frac{\partial}{\partial \chi_0^*} \langle (\hat{H} - \mu \hat{N}) \rangle_N = 0 \qquad (1)$$

$$\cancel{N}\left[-\frac{\hbar^2}{2m}\nabla^2\chi_0 + V_{ext}(\vec{r})\,\chi_0(\vec{r}) + UN\,|\chi_0(\vec{r})|^2\,\chi_0(\vec{r})\right] = \cancel{N}\mu\,\chi_0(\vec{r}) \qquad (2)$$

[Here $\mu = \frac{\delta \langle H \rangle_N}{\delta N}$ is the chemical potential.]

Rewrite this in terms of "order parameter" $\Psi(\vec{r}) = \sqrt{N}\,\chi_0(\vec{r})$: (3)

$$\boxed{\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - \mu + U\,|\Psi(\vec{r})|^2 \right]\Psi(\vec{r}) = 0} \qquad (4)$$

"Gross-Pitaevsky equation"

[with normalization: $\int d\vec{r}\, |\Psi(\vec{r})|^2 = N$]

Describes generic behavior rather well...!

Observation of Interference Between Two Bose Condensates

M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee,
D. M. Kurn, W. Ketterle, Science, 1997

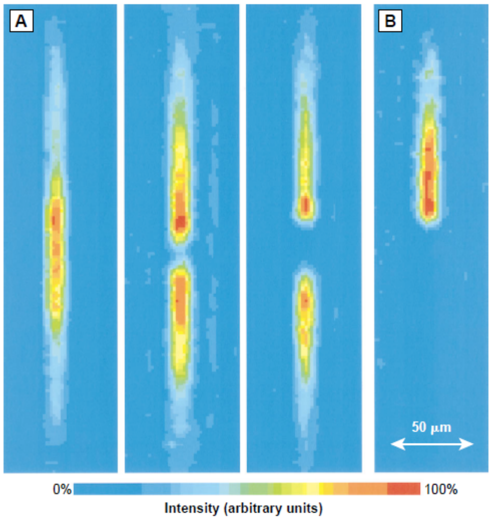
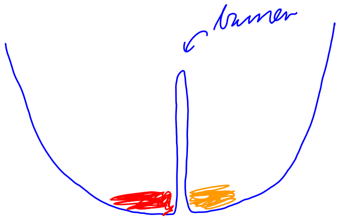


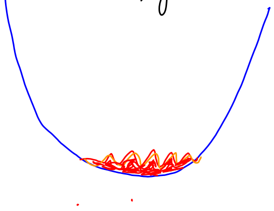
Fig. 1. (**A**) Phase-contrast images of a single Bose condensate (left) and double Bose condensates, taken in the trap. The distance between the two condensates was varied by changing the power of the argon ion laser-light sheet from 7 to 43 mW. (**B**) Phase-contrast image of an originally double condensate, with the lower condensate eliminated.

Initial configuration:



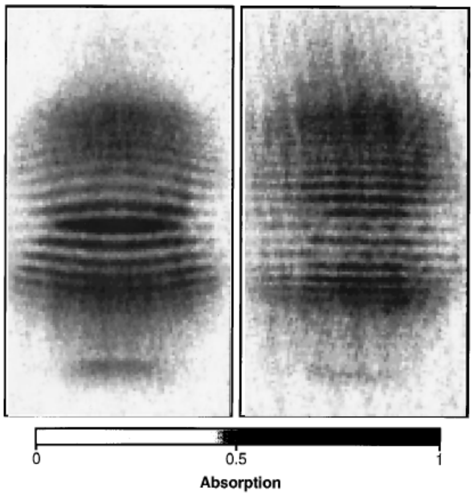
remove barrier, let two components expand & overlap.

Final configuration



BEC23

Fig. 2. Interference pattern of two expanding condensates observed after 40 ms time-of-flight, for two different powers of the argon ion laser-light sheet (raw-data images). The fringe periods were 20 and 15 μm, the powers were 3 and 5 mW, and the maximum absorptions were 90 and 50%, respectively, for the left and right images. The fields of view are 1.1 mm horizontally by 0.5 mm vertically. The horizontal widths are compressed fourfold, which enhances the effect of fringe curvature. For the determination of fringe spacing, the dark central fringe on the left was excluded.



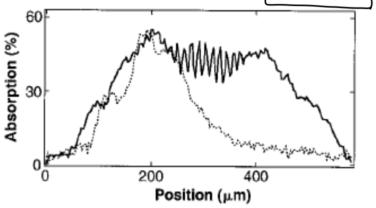


Fig. 4. Comparison between time-of-flight images for a single and double condensate, showing vertical profiles through time-of-flight pictures similar to Fig. 2. The solid line is a profile of two interfering condensates, and the dotted line is the profile of a single condensate, both released from the same double-well potential (argon ion laser power, 14 mW; fringe period, 13 μm; time of flight, 40 ms). The profiles were horizontally integrated over 450 μm. The dashed profile was multiplied by a factor of 1.5 to account for fewer atoms in the single condensate, most likely the result of loss during elimination of the second half.

$$\Psi_N(\vec r_1,\dots \vec r_{N_L};\vec r_1{}',\dots \vec r_{N_R}{}') \sim \prod_{i=1}^{N_L} |\chi_L(\vec r_i,t)|^2 \prod_{j=1}^{N_R} |\chi_R(\vec r_j{}',t)|^2$$

$N_L + N_R = N$

Initial state: $t=0$, clouds do not overlap

Time-evolved state: large t, clouds do overlap,

(but are not in a superposition!)

[but $\Psi_N \neq (\psi_L + \psi_R)^N$
which would give interference]

BEC24

Why is interference observed despite product form?

counts particles from L or R cloud

Density operator: $\overset{(14.1)}{}$
$$\hat\rho(\vec r,t) = \left(\sum_{i\in L}\delta(\vec r-\vec r_i) + \sum_{j\in R}\delta(\vec r-\vec r_j{}')\right)$$

$$\langle\hat\rho(\vec r,t)\rangle = \int d\vec r_1\dots d\vec r_{N_L}\int d\vec r_1{}'\dots d\vec r_{N_R}{}' \; \hat\rho(\vec r,t)\,|\Psi_N(\vec r_1,\dots,\vec r_{N_L};\vec r_1{}'\dots\vec r_{N_R}{}')|^2$$

$$= N_L|\chi_L(\vec r,t)|^2 + N_R|\chi_R(\vec r,t)|^2 \quad ⊛$$

predicts no interference – because it represents an average over many runs!

But: $\langle\hat\rho(\vec r,t)\,\hat\rho(\vec r',t)\rangle$ does give oscillations within a single run; relative phase from run to run is random, which is why ⊛ does not show interference!