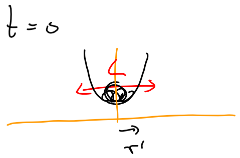
Optical Lattices

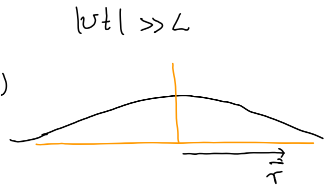
21.12.09

Time-of-flight measurements:



Let $\underset{\uparrow \text{initial state}}{P_0(\vec r', \vec p)}$ = Initial probability to find particle with velocity $\vec p$ at position $\vec r'$ (1)

$\simeq 0$ for $|\vec r'| \gg L$ = initial size of trap



$P(\vec p) = \int d\vec r \; P_0(\vec r, \vec p)$ = Initial Probability to find particle with velocity $\vec p$ (2)

After free expansion, density at $\vec r$ is:

$$\rho(\vec r, t) = \int d\vec v \int d\vec r' \; \delta\left(\vec r - \left(\vec r' + \frac{\vec p t}{m}\right)\right) P_0(\vec r', \vec p) = \int d\vec r' \; P_0\left(\vec r', m\frac{(\vec r - \vec r')}{t}\right) \quad (3)$$

$$= \overset{\vec r \gg \vec r'}{\simeq} \int d\vec r' \; P_0(\vec r', m\vec r/t) = P(\vec v = m\vec r/t) \quad (4)$$

$\Rightarrow$ After expanding for time t, density at $\vec r$ gives initial probability to find velocity $\vec v = m\vec r/t$

Quantum-mechanical calculation of $P(\vec p)$:

Consider single-particle state:

$$|\chi\rangle_1 = \underbrace{\int d\vec p \; |\vec p\rangle}_{\mathbb{1}} \underbrace{\langle \vec p | \chi\rangle_1}_{\tilde\chi(\vec p)} = \int d\vec p \; \tilde\chi(\vec p) \, |\vec p\rangle \quad (1)$$

Probability to find $\vec p$: $\quad P(\vec p) = |\tilde\chi(\vec p)|^2 = \int d\vec p_1 \; \delta(\vec p - \vec p_1) \, |\tilde\chi(\vec p_1)|^2 \quad$ (2)

$\tilde\chi(p) = \langle \vec p | \chi\rangle_1$ = wave-function in momentum representation (3)

$$= \int d\vec r \; \underbrace{\langle \vec p | \vec r\rangle}_{\frac{1}{(2\pi)^{3/2}} e^{-i\vec p\cdot\vec r}} \underbrace{\langle \vec r | \chi\rangle_1}_{\chi(\vec r)} = \int d\vec r \; e^{-i\vec p\cdot\vec r} \, \chi(\vec r) \quad (4)$$

Check normalization: $\int d\vec p \; |\tilde\chi(\vec p)| = \int d\vec r \int d\vec r' \underbrace{\int \frac{d\vec p}{(2\pi)^3} e^{i\vec p(\vec r - \vec r')}}_{\delta(\vec r - \vec r')} \chi^*(\vec r) \, \chi(\vec r')$

$$= \int d\vec r \; \chi^*(\vec r) \, \chi(\vec r) = 1 \checkmark$$

OL3

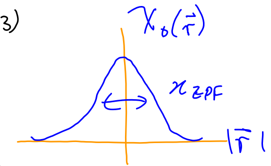
For many-body BEC-like state $|\chi\rangle_N = (|\chi\rangle_1)^{\otimes N}$ (1)

Wave function: $\tilde{\chi}_N(\vec{p}_1, \dots, \vec{p}_N) = \langle \vec{p}_1, \dots \vec{p}_N | \chi\rangle^{\otimes N}$ (2)

(symmetric under $\vec{p}_i \leftrightarrow \vec{p}_j$)

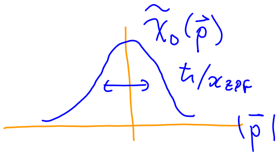
$$= \prod_{j=1}^{N} \tilde{\chi}(\vec{p}_j) \qquad (3)$$

ground state wave function

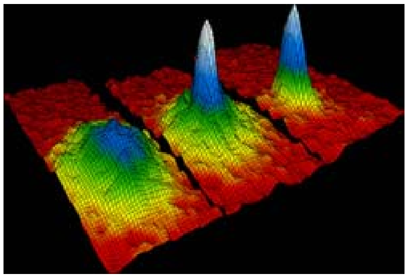


where $\tilde{\chi}_0(\vec{p}) = \int d\vec{r}\, e^{-i\vec{p}\cdot\vec{r}} \chi_0(\vec{r})$ (4)

$$P(\vec{p}) = \int d\vec{p}_1 \int d\vec{p}_2 \dots \int d\vec{p}_N \sum_{j=1}^{N} \delta(\vec{p}-\vec{p}_j) \, |\tilde{\chi}_N(\vec{p}_1, \vec{p}_2, \dots \vec{p}_N)|^2 \qquad (5)$$



$$\Longrightarrow \boxed{P(\vec{p}) = N|\tilde{\chi}(\vec{p})|^2} \qquad (6)$$



$\Longrightarrow$ time-of-flight measurement of

$\vec{\rho}(\vec{r},t) = P(m\frac{\vec{r}}{t})$ literally

gives an image of $|\text{wave-function}|^2$ in momentum space!

Second-quantized notation for momentum distribution

OL4

Consider a general single-particle state; decomposed in momentum basis:

$$|\chi\rangle_1 = \int d\vec{p}\, \tilde{\chi}(\vec{p})\, |\vec{p}\rangle \qquad (1)$$

$$= \int d\vec{p}\, \tilde{\chi}(\vec{p})\, \underbrace{a^\dagger_{\vec{p}} |0\rangle}_{|\vec{p}\rangle} \qquad (2)$$

Momentum probability distribution: "count" occupation of $\vec{p}$-modes!

$$\boxed{P(\vec{p}) = \langle a^\dagger_{\vec{p}} a_{\vec{p}} \rangle} \qquad (3)$$

$$= \int d\vec{p}' \int d\vec{p}''\, \chi^*(\vec{p}')\, \chi(\vec{p}'')\, \langle 0| \underbrace{a_{\vec{p}'} (a^\dagger_{\vec{p}}}_{\delta(\vec{p}'-\vec{p})} \underbrace{a_{\vec{p}})\, a^\dagger_{\vec{p}''}}_{\delta(\vec{p}-\vec{p}'')} |0\rangle \qquad (4)$$

$$= |\chi(\vec{p})|^2 \quad \checkmark . \qquad (5)$$

consistent with (2.2) ✓

Def. (4.3) also works for many-body states! $\boxed{\text{OL5}}$

As an example, consider again BEC-like state $|\chi\rangle_N \overset{(3.1)}{=} |\chi\rangle_1^{\otimes N}$ (1)

To write this state in 2nd-quantized form, rewrite single-particle state as:

$$|\chi\rangle_1 = \underbrace{\int d\vec{p}\ \chi(\vec{p})\ a^\dagger_{\vec{p}}}_{a^\dagger_\chi}\ |0\rangle \equiv a^\dagger_\chi |0\rangle \qquad (2)$$

Using $[a_{\bar{p}}, a^\dagger_{\bar{p}'}] = \delta(\bar{p}-\bar{p}')$, $\int dp |\hat{\chi}(\vec{p})|^2 = 1$ (for normalization) (3)

we have $[a_\chi, a^\dagger_\chi] = \int dp \int dp'\ \chi^*(p)\ \chi(p')\ [a^\dagger_p, a_{p'}] = \int dp\ |\chi(p)|^2 = 1$ ✓ (4)

hence $a^\dagger_\chi$ are bosonic creation operators. Hence

$|\chi\rangle_N = \frac{1}{\sqrt{N!}} (a^\dagger_\chi)^N |0\rangle = |N_\chi\rangle$ (notation means: N modes of type χ) (5)

↑ standard normalization factor for bosons, such that ${}_N\langle\chi|\chi\rangle_N = 1$.

Reminder: Normalization of bosonic number eigenstates: $\boxed{\text{OL6}}$

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \qquad (1)$$

check normalization and counting operator, using $[a, a^{\dagger n}] = n\, a^{\dagger n-1}$ (2)

Check #1: (norm)

$$\langle n|n\rangle = \frac{1}{n!} \langle 0| a^n a^{\dagger n} |0\rangle \qquad (3)$$

$$= \frac{1}{\cancel{n}(n-1)!} \langle 0| a^{n-1} \underbrace{\cancel{n}\, a^{\dagger n-1}}_{\text{from (2)}} |0\rangle \overset{\text{iterate}}{=} \dots = 1 \qquad (4)$$

Check #2: (number operator):

$$a|n\rangle = \frac{1}{\sqrt{n!}}\, a\, a^{\dagger n} |0\rangle \overset{(2)}{=} \frac{n}{\sqrt{n!}} (a^\dagger)^{n-1} |0\rangle = \sqrt{n}\, |n-1\rangle \qquad (5)$$

$$\Longrightarrow \langle n| a^\dagger a |n\rangle = (\sqrt{n})^2 \langle n-1|n-1\rangle = n \ ✓. \qquad (6)$$

Momentum distribution of $|\chi\rangle_N$:

$$P(\vec{p}) = {}_N\langle \chi | a^\dagger_{\vec{p}} a_{\vec{p}} | \chi \rangle_N \quad = ? \qquad (1)$$

But

$$a_{\vec{p}} |\chi_N\rangle = a_{\vec{p}} \frac{1}{\sqrt{N!}} (a^\dagger_\chi)^N |0\rangle \qquad (2)$$

but

$$[a_{\vec{p}}, a^\dagger_\chi] \overset{(5.2)}{=} [a_{\vec{p}}, \int d\vec{p}\,' \chi(\vec{p}\,') a^\dagger_{\vec{p}'}] = \chi(p) \qquad (3)$$

$$\Rightarrow \quad [a_{\vec{p}}, (a^\dagger_\chi)^N] = (a^\dagger_\chi)^{N-1} N \chi(\vec{p}) \qquad (4)$$

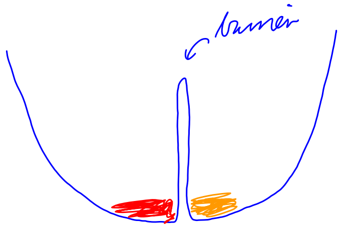
so

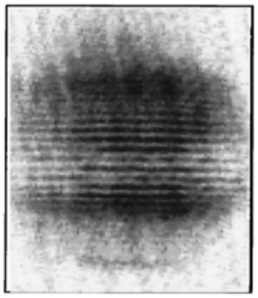
$$a_{\vec{p}} |\chi_N\rangle = \frac{N}{\sqrt{N}} \chi(\vec{p}) \frac{(a^\dagger_\chi)^{N-1}}{\sqrt{(N-1)!}} |0\rangle = \sqrt{N}\, \chi(\vec{p})\, |\chi\rangle_{N-1} \qquad (5)$$

$$\Rightarrow \quad \boxed{P(\vec{p})} \overset{(1)}{=} {}_{N-1}\langle \chi | \chi^*(\vec{p}) \sqrt{N} \sqrt{N} \chi(\vec{p}) | \chi \rangle_{N-1} = \boxed{N |\chi(\vec{p})|^2} \checkmark \qquad (6)$$

consistent with (3.6)

Interference experiments:





$$S\, \psi_L(p)\, \psi_R(p') = \psi_L(p) \psi_R(p') + \psi_L(p') \psi_R(p)$$

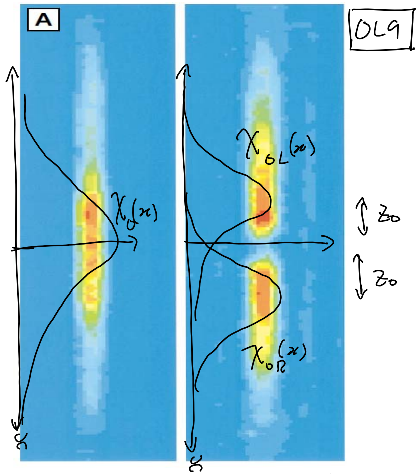
Initial wavefunction in momentum space; symmetrization operator:

$$\tilde{\Psi}_N(\vec{p}_1, \dots \vec{p}_{N_L}; \vec{p}\,'_1, \dots \vec{p}\,'_{N_R}) \sim S \prod_{i=1}^{N_L} \chi_{0L}(\vec{p}_i, t) \prod_{j=1}^{N_R} \chi_{0R}(\vec{p}\,'_j, t)$$

$$\langle \rho(\vec{r}, t) \rho(\vec{r}\,', t) \rangle = P(\vec{p}_1 = \frac{m\vec{r}}{t}; \vec{p}_2 = \frac{m\vec{r}\,'}{t})$$

joint probability distribution to find one particle with momentum $\vec{p}_1$ and another with momentum $\vec{p}_2$

$$= \int d\vec{p}_1 \dots \int d\vec{p}_N \sum_{i=1}^{N} \delta(\vec{p} - \vec{p}_i) \sum_{j=1}^{N} \delta(\vec{p} - \vec{p}_j) \, |\tilde{\Psi}_N(\vec{p}_1, \vec{p}_2, \vec{p}_3 \dots \vec{p}_N)|^2$$



before splitting

after splitting condensate

Consider 1D case, shifted wave functions

$$\left.\begin{matrix}|\chi_{0L}\rangle \\ |\chi_{0R}\rangle\end{matrix}\right\} = \left\{\begin{matrix}e^{i\varphi_L} \\ e^{i\varphi_R}\end{matrix}\right\} \underbrace{e^{i\hat{p}_z(\mp x_0)}}_{\text{shift operator}} |\chi_0\rangle$$

$\uparrow$ random overall phases

$$\tilde{\chi}_{\substack{0L \\ (R)}}(p) = \langle p | \chi_{\substack{0L \\ (R)}} \rangle$$

$$= e^{i\varphi_{\substack{L \\ (R)}}} e^{\overset{(+)}{-} i p x_0/2} \underbrace{\langle p | \chi_0 \rangle}_{\tilde{\chi}_0(p)}$$

Consider 2-atom system (for simplicity):

$$\tilde{\chi}_2(p, p') = \mathcal{S}\left(e^{i\varphi_L} e^{-ipz_0} \tilde{\chi}_0(p)\right)\left(e^{i\varphi_R} e^{+ip'z_0} \tilde{\chi}_0(p')\right)$$

$$= e^{i(\varphi_L+\varphi_R)} \tilde{\chi}_0(p) \tilde{\chi}_0(p') e^{-iz_0(p-p')} + p \leftrightarrow p'$$

$\hookleftarrow$ effect of $\mathcal{S}$, due to indistinguishability!

$$= e^{i(\varphi_L+\varphi_R)} \tilde{\chi}_0(p) \tilde{\chi}_0(p') \, 2\cos z_0(p-p')$$

Probability to find particle with momentum p:

$$P(p) = \int dp' \, |\chi_2(p,p')|^2$$

$$= \int dp' \, |\tilde{\chi}_0(p)|^2 |\tilde{\chi}_0(p')|^2 \, |\underbrace{e^{-iz_0(p-p')} + e^{iz_0(p-p')}|^2}_{(2 + 2\cos 2z_0(p-p'))}$$

$p' - p \equiv p''$

shift:
$$= \int dp'' \, |\tilde{\chi}_0(p)|^2 |\tilde{\chi}_0(p''-p)|^2 \cdot 4 = \text{independent of } z_0.$$

Ensemble average produces no interference fringes!

But: $P(p, p+\delta) = |\tilde{\chi}_2(p, p+\delta)|^2 = |\chi_0(p)|^2 |\chi(p+\delta)|^2 \, 2\cos^2(z_0\delta)$

Joint distribution for p and $p+\delta$ produces interference fringes!!

$\hookrightarrow$ due to indistinguishability!

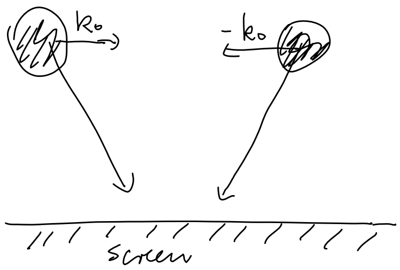
But: this analysis does not generalize easily, since $\mathcal{S}$ is complicated for $N_L + N_R$. So, rather use 2nd quantized language.

2nd-quantized analysis of interference

OL11

Consider 1-dimensional x, k :

$$|g\rangle = \left(e^{i\varphi_1} a^\dagger_{k_0}\right)^{N_1} \left(e^{i\varphi_2} a^\dagger_{-k_0}\right)^{N_2} |0\rangle = e^{i\varphi_1 N_1} e^{i\varphi_2 N_2} |(N_1)_{k_0}; (N_2)_{-k_0}\rangle$$



$$\hat\psi(x) = \sum_k e^{-ikx} a_k$$

$$\langle \hat N(x) \rangle = \langle \hat\psi^\dagger(x) \hat\psi(x) \rangle$$

$$= \sum_{kk'} e^{i(k-k')x} \langle a^\dagger_k a_{k'} \rangle$$

phases cancel!

$$\langle a^\dagger_k a_{k'} \rangle = \langle (N_1)_{k_0}; (N_2)_{-k_0} | \underbrace{a^\dagger_k a_{k'}} | (N_1)_{k_0}; (N_2)_{-k_0} \rangle$$

$$= \delta_{kk'} \left(\delta_{k,k_0} N_1 + \delta_{k,-k_0} N_2 \right)$$

$$\Rightarrow \quad \langle \hat N(x) \rangle = N_1 + N_2 \quad \Rightarrow \text{ no oscillations!}$$

Density correlations:

OL12

$$C(x,\Delta) \equiv \langle \hat N(x) \hat N(x+\Delta) \rangle$$

$$= \langle \hat\psi^\dagger(x) \hat\psi(x) \hat\psi(x+\Delta) \psi(x+\Delta) \rangle$$

$$= \sum_{kk'\bar k \bar k'} e^{ix(k-k')} e^{i(x+\Delta)(\bar k - \bar k')}$$

$$\times \langle (N_1)_{k_0}; (N_2)_{-k_0} | a^\dagger_k a_{k'} a^\dagger_{\bar k} a_{\bar k'} | (N_1)_{k_0}; (N_2)_{-k_0} \rangle$$

$$① \quad \underbrace{a^\dagger_k a_{k'}} \, \underbrace{a^\dagger_{\bar k} a_{\bar k'}}$$

$$② \quad \underbrace{a^\dagger_k \underbrace{a_{k'} a^\dagger_{\bar k}} a_{\bar k'}}$$

$$\equiv C_1 + C_2$$

$$① = \delta_{kk'} \left(\delta_{k,k_0} N_1 + \delta_{k,-k_0} N_2 \right) \delta_{\bar k, \bar k'} \left(\delta_{\bar k, k_0} N_1 + \delta_{\bar k, -k_0} N_2 \right)$$

$$C_1 = \sum_k e^{ix\cdot 0} \left(\delta_{k,k_0} N_1 + \delta_{k,-k_0} N_2 \right) \sum_{\bar k} e^{i(x+\Delta)\cdot 0} \left(\delta_{\bar k, k_0} N_1 + \delta_{\bar k, -k_0} N_2 \right)$$

$$= (N_1 + N_2)^2$$

neglect for $N_1, N_2 \gg 1$

$$② = \delta_{k'\bar{k}} \left(\delta_{\bar{k},k_0}(N_1+1) + \delta_{\bar{k},-k_0}(N_2+1) \right) \delta_{k\bar{k}'} \left(\delta_{k,k_0} N_1 + \delta_{k,-k_0} N_2 \right)$$

$$C_2 = \sum_{k\bar{k}} \underbrace{e^{ix(k-\overset{\bar{k}}{\cancel{k'}})} e^{i(x+\Delta)(\bar{k}-\overset{k}{\cancel{\bar{k}'}})}}_{e^{ix(\underbrace{k-\bar{k}+\bar{k}-k}_{=0})} \, e^{i\Delta(\bar{k}-k)}} \Big(\underset{a}{\delta_{\bar{k},k_0} N_1} + \underset{b}{\delta_{\bar{k},-k_0} N_2} \Big) \Big(\underset{c}{\delta_{k,k_0} N_1} + \underset{d}{\delta_{k,-k_0} N_2} \Big)$$

$$= \overset{ac}{N_1^2 \, e^{i\Delta(k_0-k_0)}} + \overset{bd}{N_2^2 \, e^{i\Delta(-k_0+k_0)}} + N_1 N_2 \Big(\overset{ad}{e^{i\Delta(k_0+k_0)}} + e^{i\Delta(-k_0-k_0)} \Big)$$

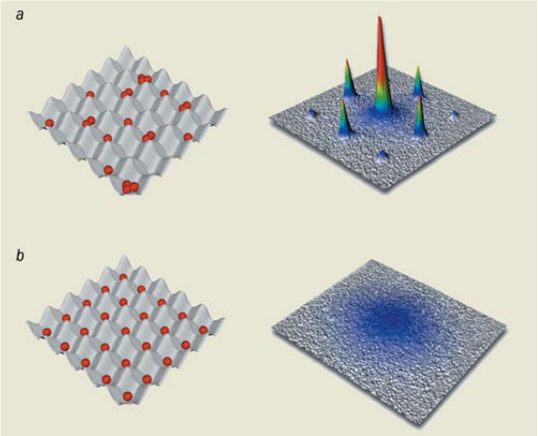
$$= N_1^2 + N_2^2 + 2 N_1 N_2 \cos 2\Delta k_0 \implies \text{interference !!}$$

Optical lattices: Superfluid-Mott Transition

Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms

Markus Greiner*, Olaf Mandel*, Tilman Esslinger†, Theodor W. Hänsch* & Immanuel Bloch*

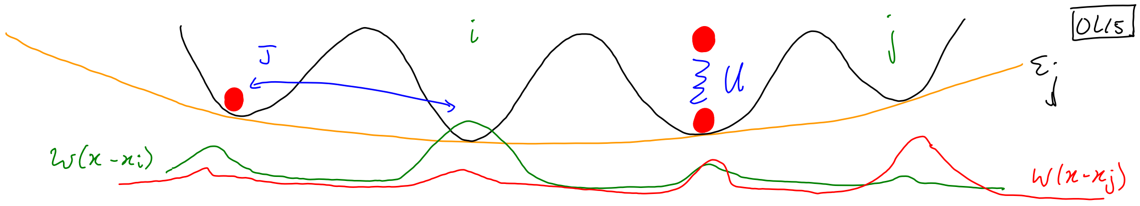
Nature, 2002



150 000 lattice sites

2.5 atoms per site at center of trap

For a system at a temperature of absolute zero, all thermal fluctuations are frozen out, while quantum fluctuations prevail. These microscopic quantum fluctuations can induce a macroscopic phase transition in the ground state of a many-body system when the relative strength of two competing energy terms is varied across a critical value. Here we observe such a quantum phase transition in a Bose–Einstein condensate with repulsive interactions, held in a three-dimensional optical lattice potential. As the potential depth of the lattice is increased, a transition is observed from a superfluid to a Mott insulator phase. In the superfluid phase, each atom is spread out over the entire lattice, with long-range phase coherence. But in the insulating phase, exact numbers of atoms are localized at individual lattice sites, with no phase coherence across the lattice; this phase is characterized by a gap in the excitation spectrum. We can induce reversible changes between the two ground states of the system.



$$H = -J\sum_{\langle i,j\rangle}\hat{a}_i^\dagger\hat{a}_j + \sum_i \epsilon_i\hat{n}_i + \frac{1}{2}U\sum_i \hat{n}_i(\hat{n}_i - 1)$$

$$J = -\int d^3x\, w(\mathbf{x}-\mathbf{x}_i)(-\hbar^2\nabla^2/2m + V_{\text{lat}}(\mathbf{x}))w(\mathbf{x}-\mathbf{x}_j) \qquad U = (4\pi\hbar^2 a/m)\int|w(\mathbf{x})|^4 d^3x$$

$J \gg U$: $|\Psi_{\text{SF}}\rangle_{U=0} \propto \left(\sum_{i=1}^{M}\hat{a}_i^\dagger\right)^N|0\rangle$

Superfluid

$J \ll U$: $|\Psi_{\text{MI}}\rangle_{J=0} \propto \prod_{i=1}^{M}(\hat{a}_i^\dagger)^n|0\rangle$ M = lattice sites

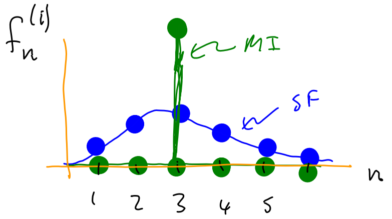
Mott-Insulator

for $N = L\cdot M$

Interpolate with Gutzwiller-Ansatz:

$$|\psi_{GW}\rangle = \prod_{i=1}^{M}\Big(\underbrace{\sum_{n=0}^{\infty} f_n^{(i)}|n\rangle_i}_{|\phi_i\rangle}\Big)$$

Fock state with n atoms on site i



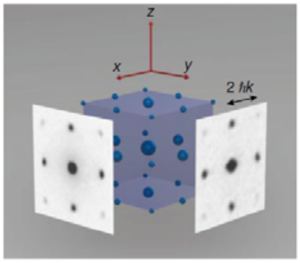


Figure 1 Schematic three-dimensional interference pattern with measured absorption images taken along two orthogonal directions. The absorption images were obtained after ballistic expansion from a lattice with a potential depth of $V_0 = 10E_r$ and a time of flight of 15 ms.

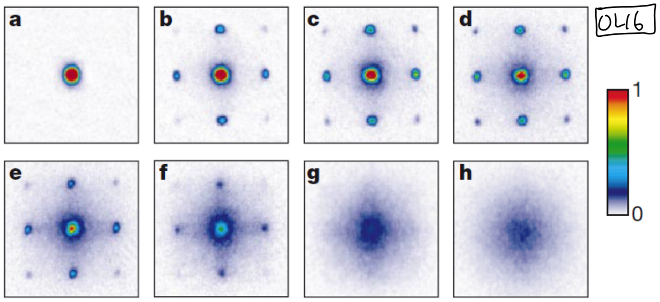


Figure 2 Absorption images of multiple matter wave interference patterns. These were obtained after suddenly releasing the atoms from an optical lattice potential with different potential depths V_0 after a time of flight of 15 ms. Values of V_0 were: **a**, 0 E_r; **b**, 3 E_r; **c**, 7 E_r; **d**, 10 E_r; **e**, 13 E_r; **f**, 14 E_r; **g**, 16 E_r; and **h**, 20 E_r.

Interference depends on: (1D argument)

$$\rho(\vec{k}) = \sum_{ij}\langle a_i^\dagger a_j\rangle e^{i\vec{k}\cdot(\vec{r}_i - \vec{r}_j)}$$

$$\langle a_i^\dagger a_j\rangle_{SF} \simeq \text{const.} \Rightarrow \rho_{SF}(\vec{k}) \simeq \sum_{ij} e^{i\vec{k}\cdot(\vec{r}_i-\vec{r}_j)} = \begin{cases}\text{large if } \vec{k} \in \text{reciprocal lattice}\\ 0 \quad \text{otherwise}\end{cases}$$

$$\langle a_i^\dagger a_j\rangle_{MI} \simeq \delta_{ij} \Rightarrow \rho_{MI}(\vec{k}) \simeq \sum_i e^{i\vec{k}\cdot 0} \simeq \text{no structure in } \vec{k}\text{-space.}$$

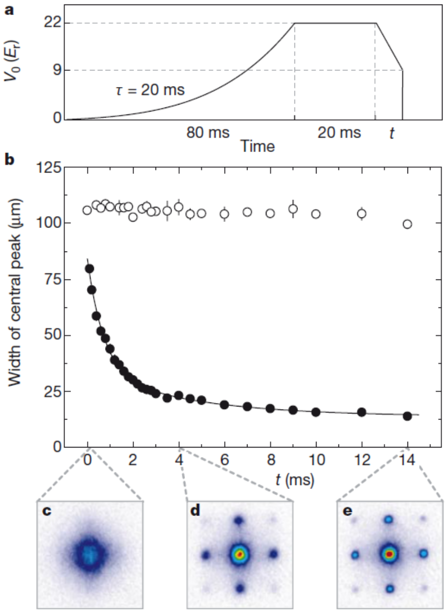


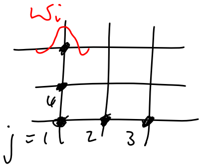
Figure 3 Restoring coherence. **a**, Experimental sequence used to measure the restoration of coherence after bringing the system into the Mott insulator phase at $V_0 = 22E_r$ and lowering the potential afterwards to $V_0 = 9E_r$, where the system is superfluid again. The atoms are first held at the maximum potential depth V_0 for 20 ms, and then the lattice potential is decreased to a potential depth of 9 E_r in a time *t* after which the interference pattern of the atoms is measured by suddenly releasing them from the trapping potential. **b**, Width of the central interference peak for different ramp-down times *t*, based on a lorentzian fit. In case of a Mott insulator state (filled circles) coherence is rapidly restored already after 4 ms. The solid line is a fit using a double exponential decay ($\tau_1 = 0.94(7)$ ms, $\tau_2 = 10(5)$ ms). For a phase incoherent state (open circles) using the same experimental sequence, no interference pattern reappears again, even for ramp-down times *t* of up to 400 ms. We find that phase incoherent states are formed by applying a magnetic field gradient over a time of 10 ms during the ramp-up period, when the system is still superfluid. This leads to a dephasing of the condensate wavefunction due to the nonlinear interactions in the system. **c–e**, Absorption images of the interference patterns coming from a Mott insulator phase after ramp-down times *t* of 0.1 ms (**c**), 4 ms (**d**), and 14 ms (**e**).

$$|\psi_{PI}\rangle = \prod_{i=1}^{M} (e^{i\phi_i} a_i^\dagger)^n$$

Phase-incoherent

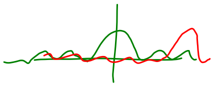
random local phases

Superfluid Ground state for $J \gg U$;



Ansatz: $|\psi_{SF}\rangle \simeq |\chi\rangle_1^{\otimes N}$

with $\chi(\vec{x}) = \langle \vec{x} | \chi \rangle = \frac{1}{L^{1/2}} \sum_j w(\vec{x} - \vec{r}_j)$ Wannier functions



particle is created with equal probability on all $L \equiv$ # of lattice sites

normalization: $$\int d\vec{x}\, |\chi(\vec{x})|^2 = \sum_{jj'} \frac{1}{L} \underbrace{\int d\vec{x}\; w(\vec{x} - \vec{r}_j)\, w^*(\vec{x} - \vec{r}_{j'})}_{= \delta_{jj'}} = 1$$

Property of Wannier functions

In momentum space:

$$\tilde{\chi}(\vec{p}) = \int d\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, \chi(\vec{x}) = \frac{1}{L^{1/2}} \sum_j \int d\vec{x}\; e^{-i\vec{p}\cdot\vec{x}}\; w(\vec{x} - \vec{r}_j)$$

shift: $\vec{x} = \vec{y} + \vec{r}_j$ $$= \frac{1}{L^{1/2}} \sum_j e^{-i\vec{p}\cdot\vec{r}_j} \underbrace{\int d\vec{y}\; e^{-i\vec{p}\cdot\vec{y}}\, w(\vec{y})}$$

only weakly dependent on $\vec{p}$ if $w(\vec{y})$ is peaked sharply

$$\tilde{\chi}(\vec{p}) \simeq \begin{cases} 1 & \text{if } \vec{p}\cdot\vec{r}_j = 1 \;\forall\; \vec{r}_j \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow \vec{p} \in$ reciprocal lattice vector! $\Rightarrow$ discrete peaks!

Alternative view: $P_{SF}(\vec{p}) \simeq \langle SF | a^\dagger_{\vec{p}} a_{\vec{p}} | SF \rangle$

$a_{\vec{p}} = \frac{1}{L^{1/2}} \sum_j e^{-i\vec{p}\cdot\vec{r}} a_j$

$$= \frac{1}{L} \sum_{ij} e^{i\vec{p}\cdot(\vec{r}_i - \vec{r}_j)} \underbrace{\langle SF | a_i^\dagger a_j | SF \rangle}_{\simeq \text{ independent of } i \text{ and } j}$$

$\Rightarrow$ peaked at $\vec{p} \in$ reciprocal lattice vector

Mott ground state for $J \ll U$:

$$|MI\rangle = \prod_{i=1}^{L} \left(a_i^\dagger\right)^n |0\rangle$$

(n atoms per lattice site)

$$P_{MI}(\vec{p}) = \langle MI | a^\dagger_{\vec{p}} a_{\vec{p}} | MI \rangle$$

$$\tilde{\psi}(\vec{p}) = \sum_i e^{-i\vec{p}\cdot\vec{r}_i} a_i \cdot r_i$$

$$= \frac{1}{L} \sum_{ij} e^{i\vec{p}\cdot(\vec{r}_i - \vec{r}_j)} \underbrace{\langle MI | a_i^\dagger a_j | MI \rangle}_{\delta_{ij}}$$

$\simeq$ independent of $\vec{p}$! $\Longrightarrow$ blurred pattern