CHSH inequality (Clauser, Horne, Shimony, Holt, 1969) PRL 23, 880 (1969)

Bell 17

Same physical setup as considered by Bell, see page Bell 2, but with 4 different possible dector polarizations:

$\hat{a}$ and $\hat{a}'$ for Alice, $\hat{b}$ and $\hat{b}'$ for Bob.

define $$\boxed{S = E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') + E(\hat{a}',\hat{b}) + E(\hat{a}',\hat{b}')}$$

where $E(\hat{a},\hat{b}) = \langle 0 | \sigma_{1a}\, \sigma_{2b} | 0\rangle$ as in (13.1)

is statistical average over many repitions of experiment, with with fixed $\hat{a}, \hat{b}$.

Classical reasoning leads to: $\boxed{|S| \leq 2 \qquad \text{(CHSH inequality)}}$

Quantum Mechanics yields $|S| > 2$ for some choices of $\hat{a}, \hat{a}', \hat{b}, \hat{b}'$.

Derivation of CHSH inequality assuming hidden variables:

Bell 18

$$E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') = \int d\lambda\, w(\lambda) \left[\tau_{1a}\tau_{2b} - \tau_{1a}\tau_{2b'} \right]$$

$$= \int d\lambda\, w(\lambda)\, \tau_{1a}\tau_{2b} \left[1 \pm \tau_{1a'}\tau_{2b'} \right]$$

$$- \int d\lambda\, w(\lambda)\, \tau_{1a}\tau_{2b'} \left[1 \pm \tau_{1a'}\tau_{2b} \right]$$

cancel

[$\pm$ means: equality holds for both choices of sign]

$$|E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}')| \qquad \left[\text{use } \left|\int d\lambda\, f(\lambda)\right| \leq \int d\lambda\, |f(\lambda)| \right]$$

$$\leq \int d\lambda\, w(\lambda) \underbrace{|\tau_{1a}\tau_{2b}|}_{=1} \underbrace{\left|\left[1 \pm \tau_{1a'}\tau_{2b'}\right]\right|}_{\geq 0} + \int d\lambda\, w(\lambda) \underbrace{|\tau_{1a}\tau_{2b'}|}_{=1} \underbrace{\left|\left[1 \pm \tau_{1a'}\tau_{2b}\right]\right|}_{\geq 0}$$

$$= \int d\lambda\, w(\lambda) \left[2 \pm \left(\tau_{1a'}\tau_{2b'} + \tau_{1a'}\tau_{2b}\right)\right] = 2 \pm \left(E(a',b') + E(a',b) \right)$$

This inequality includes the case: (choose upper/lower sign on left/right

$$\Rightarrow -\left(2 + E(a',b') + E(a',b)\right) \leq E(\hat{a},\hat{b}) - E(\hat{a},\hat{b}') \leq 2 - E(a',b') + E(a',b))$$

$$\Rightarrow \quad -2 \leq S \leq 2 . \qquad \text{(CSHS inequality)} \qquad \square$$

Bell 19

Simple interpretation: with hidden variable interpretation, there are predetermined results (depending on λ) for each combination of detector orientations:

$$A(\hat{a}), \quad A(\hat{a}'), \quad B(\hat{b}), \quad B(\hat{b}') \qquad (1)$$

$$\tau_{1a}(\lambda), \quad \tau_{1a'}(\lambda), \quad \tau_{2b}(\lambda), \quad \tau_{2b'}(\lambda), \qquad \text{all} \in \{+1, -1\}. \qquad (2)$$

then for that run,

$$S(\lambda) = \tau_{1a}\tau_{2b} - \tau_{1a}\tau_{2b'} + \tau_{1a'}\tau_{2b} + \tau_{2a'}\tau_{2b'} \qquad (3)$$

$$= \underbrace{\tau_{1a}}_{1 \text{ or } -1}\underbrace{(\tau_{2b} - \tau_{2b'})}_{= 0 \text{ or } 2} + \underbrace{\tau_{1a'}}_{1 \text{ or } -1}\underbrace{(\tau_{2b} + \tau_{2b'})}_{= 2 \text{ or } 0} = \pm 2 \qquad (4)$$

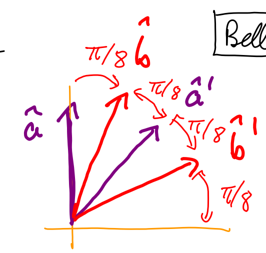
averaging many runs, each of which gives ± 2, we would get

$$\boxed{-2 < S < 2} \quad (5) \qquad \text{for the average,} \qquad S = \int d\lambda \; w(\lambda)\, S(\lambda).$$

The fact that QM violates this inequality (see below) means that we cannot assign

Bell 20

Quantum Mechanical Prediction for all vectors in same plane, with specified angles w.r.t. $\hat{z}$:



$$\theta_a = 0, \quad \theta_{a'} = \pi/4, \quad \theta_b = \pi/8, \quad \theta_{b'} = 3\pi/8.$$

$$E_{QM}(\hat{\ell}, \hat{n}) = -\cos(\theta_\ell - \theta_n)$$

$$\Rightarrow \quad E_{QM}(\hat{a}, \hat{b}) = -\cos \pi/8 \qquad E_{QM}(\hat{a}, \hat{b}') = -\cos 3\pi/8 = -\sin \pi/8$$

$$E_{QM}(\hat{a}', \hat{b}) = -\cos \pi/8 \qquad E_{QM}(\hat{a}', \hat{b}') = -\cos \pi/8$$

$$|S_{QM}| = |-\cos\pi/8 + \cos 3\pi/8 - \cos\pi/8 - \cos\pi/8|$$

$$= 2.388 \quad > 2 \quad !! \qquad \text{Violates} \quad |S| < 2.$$

Such violations were demonstrated by Aspect, 1981 – 1982.

Greenberger-Horne-Zeilinger (GHZ) equation

Needs 3 particles, gives a <u>stronger</u> statement of incorrectness of local realism, since it involves an <u>equality</u> (instead of inequality), hence makes a fully <u>deterministic</u> statement!

Consider $$|\psi\rangle_3 = \frac{1}{\sqrt{2}}\left(|\uparrow_1 \uparrow_2 \uparrow_3\rangle - |\downarrow_1 \downarrow_2 \downarrow_3\rangle\right) \qquad (1)$$ "GHZ state"

Since $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\quad \sigma_x|\uparrow\rangle = |\downarrow\rangle$, $\quad \sigma_x|\downarrow\rangle = |\uparrow\rangle$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\quad \sigma_y|\uparrow\rangle = i|\downarrow\rangle$, $\quad \sigma_y|\downarrow\rangle = -i|\uparrow\rangle$

$|\psi\rangle_3$ is an eigenstate of

$\sigma_{1x}\sigma_{2y}\sigma_{3y}$ with eigenvalue $+1$

$\sigma_{1y}\sigma_{2x}\sigma_{3y}$ $\quad +1$

$\sigma_{1y}\sigma_{2y}\sigma_{3x}$ $\quad +1$

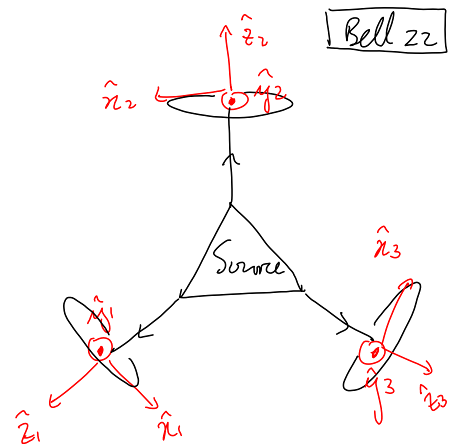
$$1 = \sigma_{1x}\sigma_{2y}\sigma_{3y}\;\sigma_{1y}\sigma_{2x}\sigma_{3y}\;\sigma_{1y}\sigma_{2y}\sigma_{3x} = -\sigma_{1x}\sigma_{2x}\sigma_{3x} \qquad +1$$

For each particle, define

$\hat{z}$ - direction along its direction of flight,

$\hat{x}$ - direction in common plane of flight

$\hat{y}$ - direction $\perp$ to common plane of flight.



So, if measurement of σ_{1y} gives 1 and σ_{2y} gives $+1$ (or -1) then we can predict with certainty that σ_{3x} must give $+1$ (or -1).

Similarly, we can predict both σ_{ix} and σ_{iy} for any i by making appropriate measurements on other two particles.

Local realism a la EPR would then imply that there must be $\boxed{\text{Bell 23}}$ hidden variables λ, with $\tau_{ix}(\lambda) \in \{1,-1\}$ and $\tau_{iy}(\lambda) \in \{1,-1\}$ being functions of λ, correlated in such a way that for any given λ:

$$\tau_{1x}\,\tau_{2y}\,\tau_{3y} = 1 \qquad (1)$$
$$\tau_{1y}\,\tau_{2x}\,\tau_{3y} = 1 \qquad (2) \qquad \underline{\text{and}} \qquad \tau_{1x}\,\tau_{2x}\,\tau_{3x} = -1 \qquad (4)$$
$$\tau_{1y}\,\tau_{2y}\,\tau_{3x} = 1 \qquad (3)$$

But this is impossible, since from $(1)\cdot(2)\cdot(3)$ we obtain

$$\tau_{1x}\tau_{2x}\tau_{3x}\underbrace{\left(\tau_{1y}\tau_{2y}\tau_{3y}\right)^2}_{=1} = 1 \implies \tau_{1x}\tau_{2x}\tau_{3x} = 1, \qquad (5)$$

and (5) contradicts (4). Conclusion: predictions of QM (which have been confirmed experimentally!) contradict assumption of local realism!

Mermin's discussion of GHZ [Am. J. Phys., 58, 731 (1990)] $\boxed{\text{Bell 24}}$

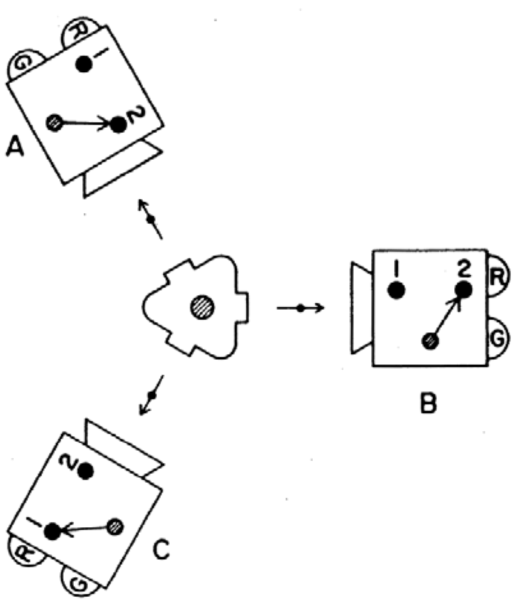


Fig. 1. The three detectors A, B, and C, viewed from above (down the x axis). Their switch settings are 221. When the button on the source in the middle is pushed, three particles (shown *en route*) emerge and move in the horizontal plane to the three detectors.

Each detector A, B, C has two possible settings, 1 or 2; when triggered, each detector flashes red (R) or green (G).

For each run of experiment:

- set ABC randomly to 122, 212, 221 or 111 "odd number of dectors is set to 1".
- record the outcomes of flashes.

Summary of results of many such experiments:
Depending on types of detector settings, outcomes are different: (1)

Type I setting: only one detector is set to 1:

122, 212, 221 } produce, with equal probability, a random choice among {RRR, Rgg, gRg, ggR}

"always odd number of reds"

Type II setting: all three detectors are set to 1: (2)

111 produces, with equal probability, a random choice among {ggg, gRR, RgR, RRg}

"never odd number of reds"

For type I setting (only one detector is set to 1) $\Rightarrow$ odd number of reds
for any detector i we can predict its outcome with certainty by observing other two ($j, k \neq i$)

if j, k = RR or gg $\Rightarrow$ i = R

if j, k = Rg or gR $\Rightarrow$ i = g

Analysis assuming local realism: for a given run, each particle must carry instructions for how its detector should flash for either of two possible switch settings: symbolize instructions by pair of letters:

$\begin{matrix} R \\ R \end{matrix}$, $\begin{matrix} R \\ g \end{matrix}$, $\begin{matrix} g \\ R \end{matrix}$, $\begin{matrix} g \\ g \end{matrix}$ $\begin{matrix} \leftarrow \text{upper} \\ \leftarrow \text{lower} \end{matrix}\Big\}$ letter: result if switch is set to $\Big\{\begin{matrix} 1 \\ 2 \end{matrix}$

(these instructions are "hidden variables")

$\boxed{\text{Bell 27}}$

For a given run, instructions for particles A B C can be summarized by a list of the form:

general notation:				example		results for various switch settings:		
A_1	B_1	C_1	‖	R G G	for switch setting →	1 2 2	2 1 2	2 2 1
A_2	B_2	C_2	‖	G R R	produces, →	R R R	G G R	G R G

This is a "legal" instruction set, since it always produces an odd number of reds.

There are only eight legal sets of instructions, which can be seen as follows: Possible entries for switch settings 1 2 2 :

general notation

A_1 — —	‖	R — —	|	G — —	|	R — —	| G — —
— B_2 C_2	‖	— R R	|	— R G	|	— G G	| — G R

(They all must have odd number of reds, in case switches are set to 122)

$\boxed{\text{Bell 28}}$

Now count ways to fill blanks in a way that would be consistent with switch setting 221:

since B_2 is already specified, we need; to ensure odd # of R's:

if $B_2 = \underline{R} \Rightarrow (A_2, C_1) = (R,R)$ or (G,G) | if $B_2 = \underline{G} \Rightarrow (A_2, C_1) = (R,G)$ or (G,R)

— — C_1	‖	R — R	G — R	|	R — G	G — G
A_2 B_2 —	‖	R $\underline{R}$ R	R $\underline{R}$ G	|	R $\underline{G}$ G	R $\underline{G}$ R

or

— — C_1	‖	R — G	G — G	|	R — R	G — R
A_2 B_2 —	‖	G $\underline{R}$ R	G $\underline{R}$ G	|	G $\underline{G}$ G	G $\underline{G}$ R

Bell 29

Final entry for B1 is fixed by setting 212 : $\begin{matrix} - & B_1 & - \\ A_2 & - & C_2 \end{matrix}$ $\Rightarrow$ odd # reds

Setting				
$\begin{matrix} - & - & C_1 \\ A_2 & B_2 & - \end{matrix}$	R R R R R R	G G R R R G	R G G R G G	G R G R G R
or $\begin{matrix} - & - & C_1 \\ A_2 & B_2 & - \end{matrix}$	R G G G R R	G R G G R G	R R R G G G	G G R G G R

This is set of all legal instructions, according to type I settings.
Note from upper row of legal instructions, $A_1 B_1 C_1$
<u>always</u> contains an <u>odd</u> number of R's.

Bell 30

But this contradicts result of type II setting:

$A_1 B_1 C_1 \Rightarrow$ <u>never</u> gives odd # of reds!!

So, single run of type II shows inconsistency of instruction sets derived from type I settings!

How does device work? $|\psi\rangle_3 = |\uparrow_1 \uparrow_2 \uparrow_3\rangle - |\downarrow_1 \downarrow_2 \downarrow_3\rangle$

setting 1 $\Rightarrow$ measure σ_x | flash red $\Rightarrow +1$
2 σ_y | green $\Rightarrow -1$

$$\left.\begin{matrix} \sigma_{1x}\,\sigma_{2y}\,\sigma_{3y} \Rightarrow \overset{\text{eigenvalue}}{1} \\ \sigma_{1y}\,\sigma_{2x}\,\sigma_{3y} \Rightarrow 1 \\ \sigma_{1y}\,\sigma_{2y}\,\sigma_{3x} \Rightarrow 1 \end{matrix}\right\} \Rightarrow \begin{matrix} \text{even \# of greens} \\ \text{odd \# of reds} \end{matrix}$$

$$\sigma_{1x}\,\sigma_{2x}\,\sigma_{3x} \Rightarrow \overset{\text{eigenvalue}}{-1} \Rightarrow$$

odd # of greens $\Rightarrow$
even # reds