1.4 Counting Electrons (CE)

$L \curvearrowright R$

4.5.2010 CE1

(# of electrons passing from L lead via nanostructure to R lead in time Δt)

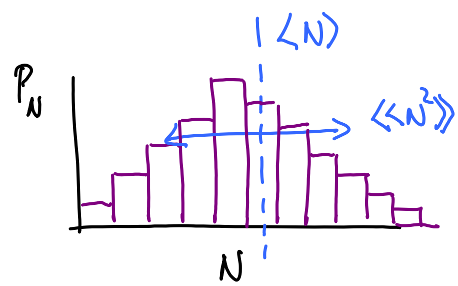
$\equiv$ (# of counts) $= N$, is a random variable. Let's study its statistics!

Do M_{tot} separate counting experiments, and note down the resulting N from each count.

Let M_N be the # of times that the count yields N. Then define:

$$P_N \equiv \lim_{M_{tot} \to \infty} \left(\frac{M_N}{M_{tot}} \right)$$

= probability that precisely N electrons pass from $L \to R$ in time Δt



CE2

Normalization: $$\sum_N P_N = 1 \quad (1)$$

Average #: $$\langle N \rangle = \sum_N N P_N \quad (2)$$

"Variance" or "Second cumulant": $$\langle\langle N^2 \rangle\rangle = \langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 \quad (3)$$

$$= \sum_N N^2 P_N - \left(\sum_N N P_N \right)^2 \quad (4)$$

Consider a useful "Fourier-representation" of the distribution P_N:

"Characteristic function": $$\Lambda(\chi) \equiv \sum_{N=0}^{\infty} P_N e^{i\chi N} = \langle e^{i\chi N} \rangle \quad (5)$$

CE3

If there are two types of events, that are statistically <u>independent</u> (say $N_\uparrow$ and $N_\downarrow$ are # of spin $\uparrow$ and spin $\downarrow$ that pass), then

"convolution"

$$P_N^{tot} = \sum_{N_\uparrow=0}^{N} P^\uparrow_{N_\uparrow} P^\downarrow_{\underbrace{N-N_\uparrow}_{N_\downarrow}} = \sum_{N_\uparrow=0}^{\infty} P^\uparrow_{N_\uparrow} \sum_{N_\downarrow=0}^{\infty} P^\downarrow_{N_\downarrow}\, \delta_{N, N_\uparrow + N_\downarrow} \qquad (1)$$

$$\Lambda^{tot}(\chi) = \sum_{N=0}^{\infty} e^{iN\chi} P_N^{tot} = \sum_{N_\uparrow=0}^{\infty} P^\uparrow_{N_\uparrow} e^{iN_\uparrow \chi} \sum_{N_\downarrow=0}^{\infty} P^\downarrow_{N_\downarrow} e^{iN_\downarrow \chi} \qquad (2)$$

$$= \Lambda^\uparrow(\chi)\, \Lambda^\downarrow(\chi) \qquad (3)$$

This is general: if different types of events (say of type $i = 1, \dots, n$) are independent, then characteristic function factorizes:

$$\boxed{\Lambda^{tot}(\chi) = \prod_{i=1}^{n} \Lambda_i(\chi)} \qquad (4)$$

CE4

Cumulants: $C_k := \frac{\partial^k}{\partial^k (i\lambda)} \ln \Lambda(\chi) \big|_{\chi=0}$ (1)

Now: $$\frac{\partial^k}{\partial^k (i\lambda)} \Lambda(\chi) = \sum_N N^k P_N e^{i\chi N} \xrightarrow{\chi=0} \langle N^k \rangle \qquad (1')$$

$$\frac{\partial}{\partial (i\chi)} \ln \Lambda(\chi) = \frac{\Lambda'(\chi)}{\Lambda(\chi)} \xrightarrow{\chi=0} \frac{\sum_N N P_N}{\sum_N P_N} = \langle N \rangle =: C_1 \qquad (2)$$

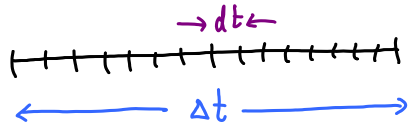
$$\frac{\partial^2}{\partial^2 (i\chi)} \ln \Lambda = \frac{\Lambda''}{\Lambda} - \left(\frac{\Lambda'}{\Lambda}\right)^2 \xrightarrow{\chi=0} \langle N^2 \rangle - \langle N \rangle^2 =: C_2 \qquad (3)$$

$$\frac{\partial^3}{\partial^3 (i\chi)} \ln \Lambda = \frac{\Lambda'''}{\Lambda} - \frac{\Lambda''\Lambda'}{\Lambda^2} - 2\frac{\Lambda'\Lambda''}{\Lambda^2} + 2\frac{(\Lambda')^3}{\Lambda^3} = \frac{\Lambda'''}{\Lambda} - 3\frac{\Lambda''\Lambda'}{\Lambda^2} + 2\frac{(\Lambda')^3}{\Lambda^3}$$

$$\xrightarrow{\chi=0} \langle N^3 \rangle - 3\langle N^2 \rangle \langle N \rangle + 2\langle N \rangle^3 =: C_3 \qquad (4)$$

Claim: all cumulants are proportional to Δt, the time of measurement. CE5

Reason: Divide $\Delta t \equiv M\,\delta t$ (1)



into M equally short intervals, dt, still long enough to contain many events each.

The events in interval i are independent from those in interval j, so:

$$\Lambda^{tot}_{\Delta t}(\chi) = \prod_{i=1}^{M} \Lambda_{dt}(\chi) \overset{\text{since all intervals are equivalent}}{=} \left[\Lambda_{dt}(\chi)\right]^M \quad (2)$$

$$\ln \Lambda^{tot}_{\Delta t}(\chi) = M \ln \Lambda_{dt}(\chi) \quad (3)$$

$$C_k(\Delta t) = \frac{\partial^k}{\partial^k(i\chi)}\left(\quad\right) = M \frac{\partial^k}{\partial^k(i\chi)}\left(\quad\right) = M\, C_k(dt) \quad (4)$$

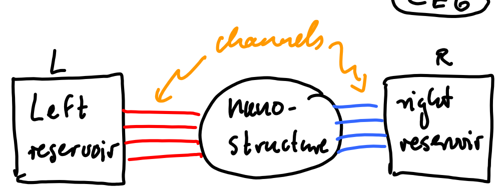
"$\frac{\Delta t}{\delta t}$" $\propto$ measurement time

1.4.1 Statistics of electron transfers

CE6

Event: transfer of electron from L to R

Count: total charge $Q = Ne$ transferred in time Δt (with $N \gg e$).



from Landauer

Average charge transferred in Δt: $\langle Q \rangle = \langle I \rangle \Delta t$ (1)

Goal: "full counting statistics", i.e. find $\Lambda(\chi)$!!

<u>Simple limit 1:</u> Tunnel junction, $T_p \ll 1 \;\forall p$

Assume electrons pass only from $L \to R$, and that transfers are uncorrelated. Divide interval: $\Delta t \overset{(5.1)}{=} M\,\delta t$, (2)

such that probability of transfer within dt is very small: $\Gamma dt \ll 1$ (3)

" " " no electrons within dt : $1 - \Gamma dt$ (4)

" two " " " : $(\Gamma dt)^2$ = negligible (5)

Short interval (CE7)

$$\Lambda_{dt}(\chi) = \langle e^{i\chi Q/e} \rangle \overset{(2.5)}{=} \underbrace{(1-\Gamma dt)\, e^{0}}_{\text{no events}} + \underbrace{\Gamma dt\, e^{i\chi}}_{\text{one event}} + \underbrace{\dots}_{\substack{\text{ignore}\\ \text{2 and more}\\ \text{events}}} \qquad (1)$$

$\Lambda(\chi) = \sum_N P_N e^{i\chi N}$

$$= 1 + \Gamma dt (e^{i\chi} - 1) \qquad (2)$$

Whole interval:

$$\Lambda_{\Delta t}(\chi) \overset{(3.4)}{=} \left(\Lambda_{dt}(\chi)\right)^M = \left(1 + \Gamma \frac{\Delta t}{M}(e^{i\chi}-1)\right)^M \qquad (3)$$

$\lim_{M\to\infty} \left(1+\frac{a}{M}\right)^M = e^a$

$$\xrightarrow{M\to\infty} \exp\left(\underbrace{\Gamma \Delta t}_{\tilde{N}} (e^{i\chi}-1)\right) \qquad (4)$$

Here $\tilde{N} = \Gamma \Delta t = \frac{\langle Q \rangle}{e}$ is average # of electrons transfered. (5)

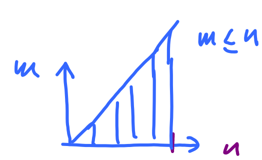
Inverse Fourier transform of (2.5), to find P_N: (CE8)

$$P_N = \int_0^{2\pi} \frac{d\chi}{2\pi}\, e^{-iN\chi}\, \Lambda(\chi) \qquad (1)$$

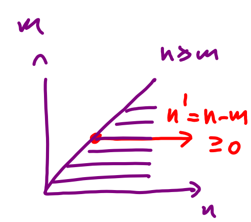
$$\overset{(7.4)}{=} e^{\tilde{N}(e^{i\chi}-1)} \overset{\text{Taylor}}{=} \sum_{n=0}^{\infty} \frac{\tilde{N}^n (e^{i\chi}-1)^n}{n!} \qquad (2)$$

Binomial expansion

$$= \sum_{n=0}^{\infty} \frac{\tilde{N}^n}{n!} \sum_{m=1}^{n} \binom{n}{m} e^{i\chi m} (-)^{n-m}$$



$$P_N \overset{(1)}{=} \sum_{m=0}^{\infty} \sum_{n'=0}^{\infty} \frac{\tilde{N}^{n'+m}}{n'!\, m!} (-)^{n'} \underbrace{\int_0^{2\pi} \frac{d\chi}{2\pi}\, e^{i\chi(m-N)}}_{\delta_{m,N}}$$



$$\boxed{P_N = \frac{\tilde{N}^N}{N!} e^{-\tilde{N}}} = \text{Poisson distribution!} \qquad (3)$$

This applies to tunnel junctions: small currents $\Rightarrow T_p \ll \ \forall p \Rightarrow$ uncorrelated tunneling events.

Simple limit 2: Ideal contact, $T_p = 1 \ \forall p$ (at zero temperature) (CE 9)

Electrons are in ideal wave states, longitudinal momentum is good quantum number, does not fluctuate. $\Rightarrow$ Current $= \sum$ momenta does not fluctuate either.

$$\Rightarrow \quad P_N = \delta_{\tilde{N}N} \qquad (1)$$

transfers are correlated ideally.

$$\Rightarrow \quad \Lambda(\chi) = \sum_N e^{i\chi N} P_N = e^{i\chi\tilde{N}} \qquad (2)$$

General case: $0 < T_p < 1$: transfers are correlated, but not ideally!

$$\boxed{\ln\Lambda(\chi) = 2_s\,\Delta t \int \frac{dE}{2\pi\hbar} \sum_p \ln\Big\{1 + T_p\left(e^{i\chi}-1\right) f_L(E)\left[1-f_R(E)\right] + T_p\left(e^{-i\chi}-1\right) f_R(E)\left[1-f_L(E)\right]\Big\}} \qquad (3)$$

"Levitov formula" (1996)

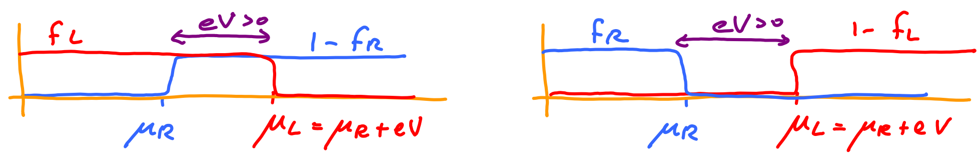
Interpretation of Levitov formula (CE 10)

- $\ln\Lambda = \sum_p \ldots \Rightarrow$ electron transfers in different channels are independent.
- $\ln\Lambda = \int dE \Rightarrow$ electron transfers at different energies are independent.
- $\ln\Lambda \neq \ldots f_L + \ldots f_R \Rightarrow$ electron transfers from $L \to R$ and $R \to L$ are correlated!

Explicitly: energies E such that $f_L = f_R = 1$, make no contribution to I, (Fermi blocking, $(1 - f_{L/R}) = 0$)

Uncorrelated transfers would produce current fluctuations.

Levitov formula for $eV \gg k_B T$ "shot noise limit" CE11



$$\ln \Lambda(\chi) \overset{(9.3)}{=} \pm \underbrace{\frac{2_s\, eV\, \Delta t}{2\pi \hbar}} \sum_p \ln\left[1 + T_p\left(e^{\pm i\chi} - 1\right)\right] \quad \text{for } eV \gtrless 0 \tag{1}$$

(1') $= \frac{G_Q V \Delta t}{e} \equiv N_{at}$ (assume to be integer) (assume >0 below)

$$= \ln \prod_p \underbrace{\left[1 - T_p + T_p e^{\pm i\chi}\right]^{N_{at}}}_{\Lambda_p(\chi) \quad (3')} \Rightarrow \Lambda(\chi) \overset{(2)}{=} \prod_p \Lambda_p(\chi) \tag{3}$$

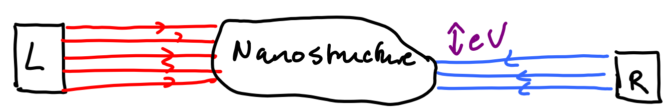
For channel p:

$$\Lambda_p(\chi) = \sum_{N=0}^{N_{at}} \underbrace{\binom{N}{N_{at}} T_p^{N} (1-T_p)^{N_{at}-N}}_{(2.5):\ P_N^{(p)}} e^{i\chi N} \tag{4}$$

$\Leftarrow$ = binomial distribution !! (5)

T_p is probability of success for one attempt, $P_N^{(p)}$ for N successes from N_{at} attempts.

For $eV > 0$, $T = 0$: CE12



Electrons from left in window $E \in [\mu_R, \mu_L]$ enter in "regular stream", with average attempt time $t_{at} = \frac{\Delta t}{N_{at}} \overset{(11.1')}{=} \frac{e}{G_Q V}$, (1)

and pass or get reflected with probability T_p or $R_p = 1 - T_p$ (2)

$$\boxed{P_N^{(p)} \overset{(5.11)}{=} \text{probability that out of } N_{at} \text{ attempts, } N \text{ pass, } N_{at}-N \text{ are reflected}}$$

Average # that passes in time Δt : $\tilde{N} = T_p N_{at} \overset{(1)}{=} T_p \frac{G_Q V \Delta t}{e}$ (3)

recalling Landauer formula for single channel $\langle I \rangle = G_Q T_p V$

$= \frac{\langle I \rangle}{e} \Delta t$ [as expected ☺] (4)

CE13

Limit of $T_p \ll 1$:

$$\Lambda_p(\chi) \overset{(11.3')}{=} \left(1 + T_p\left(e^{i\chi} - 1\right)\right)^{N_{at}} \quad (1)$$

$$\text{if } T_p \ll 1: \quad \simeq e^{\overbrace{N_{at} T_p}^{\tilde{N}\ (12.3)}\left(e^{i\chi} - 1\right)} = e^{\tilde{N}\left(e^{i\chi} - 1\right)} \overset{(7.4)}{=} \Lambda_{\text{Poisson}}(\chi) \quad (2)$$

$\Rightarrow$

$$P_N^{(p)} \overset{(11.5)}{=} \binom{N}{N_{at}} T_p^N (1 - T_p)^{N_{at} - N} \xrightarrow[\tilde{N} \ll N_{at}]{T_p \ll 1} P_N \overset{(8.3)}{=} \frac{\tilde{N}^N}{N!} e^{-\tilde{N}} \quad (3)$$

binomial — Poisson

Interpretation: for $T_p \ll 1$, regular stream incident from left, emerges as irregular stream transmitted to right! with long intervals between transmission events, which thus appear to be independent.

CE14

Cumulants: C_1 = average charge, C_2 = noise, C_3 = ...

$$C_k \overset{(4.1)}{:=} \frac{\partial^k}{\partial^k(i\lambda)} \ln \Lambda(\chi) \Big|_{\chi=0} \qquad (\text{see p. 4}) \quad (1)$$

$$C_1 \overset{(4.2)}{=} \frac{\Lambda'}{\Lambda} \quad ; \quad C_2 \overset{(4.3)}{=} \frac{\Lambda''}{\Lambda} - \left(\frac{\Lambda'}{\Lambda}\right)^2 \quad ; \quad C_3 \overset{(4.4)}{=} \frac{\Lambda'''}{\Lambda} - \frac{3\Lambda''\Lambda'}{\Lambda^2} + 2\frac{(\Lambda')^3}{\Lambda^3} \quad (2)$$

$$\ln \Lambda(\chi) \overset{(9.3)}{=} \frac{G_Q \Delta t}{e^2} \int dE \sum_p \ln\left\{1 + T_p\left(e^{i\chi} - 1\right) f_L(1 - f_R) + T_p\left(e^{-i\chi} - 1\right) f_R[1 - f_L]\right\} \quad (3)$$

Levitov

$$\partial_{i\chi} \ln\{\,\} = \frac{1}{\{\,\}}\left[T_p e^{i\chi} f_L(1 - f_R) - T_p e^{-i\chi} f_R(1 - f_L)\right] \quad (4)$$

$$\partial_{i\chi}^2 \ln\{\,\} = \frac{\left(T_p e^{i\chi} f_L(1 - f_R) + T_p e^{-i\chi} f_R(1 - f_L)\right)}{\{\,\}} - \frac{[\,]^2}{\{\,\}^2} \quad (5)$$

$$\partial_{i\chi}^3 \ln\{\,\} = \frac{T_p e^{i\chi} f_L(1 - f_R) - T_p e^{-i\chi} f_R(1 - f_L)}{\{\,\}} - 3\frac{(\,)[\,]}{\{\,\}^2} + 2\frac{[\,]^3}{\{\,\}^3} \quad (6)$$

CE15

Average charge (c_1):

$$\langle Q \rangle = e\langle N \rangle \overset{(4.2)}{=} e \left.\frac{\Lambda'}{\Lambda}\right|_{\chi=0} \tag{1}$$

Landauer ✓ ☺

$$\overset{(14.4)}{=} \Delta t \; G_Q \sum_p T_p \left[\int \frac{dE}{e} \Big[\underbrace{f_L[1-f_R] - f_R(1-f_E)}_{(f_L - f_R)} \Big] \right]_{=V} = \Delta t \langle I \rangle \tag{2}$$

Noise (c_2):

$$\langle\langle \dot{Q}^2 \rangle\rangle \equiv \langle Q^2 \rangle - \langle Q^2 \rangle = e^2 \left(\frac{\Lambda''}{\Lambda} - \left(\frac{\Lambda'}{\Lambda}\right)^2 \right) \tag{3}$$

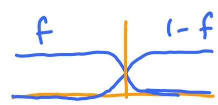
$$\overset{(14.5)}{=} G_Q \Delta t \int dE \sum_p \Big\{ T_p \big(f_L[1-f_R] + f_R[1-f_L] \big) - T_p^2 (f_L - f_R)^2 \Big\} \tag{4}$$

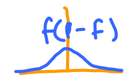
substract and add: $\quad - T_p (f_L - f_R)^2 \quad\quad + T_p (f_L - f_R)^2$

$$= G_Q \Delta t \int dE \sum_p \Big\{ T_p \big[f_L(1-f_L) + f_R(1-f_R) + T_p(1-T_p)(f_L - f_R)^2 \big] \Big\} \tag{5}$$

CE16

In equilibrium, $V = 0$:





$$f_L = f_R \equiv f(E) = \left[e^{E/k_B T} + 1 \right]^{-1} \tag{1}$$

$$\langle\langle \dot{Q}^2 \rangle\rangle \overset{(15.5)}{=} 2 \Delta t \underbrace{G_Q \sum_p T_p}_{G} \int dE \; f(1-f) \overset{(4)}{=} 2 \Delta t \, G \, k_B T \tag{2}$$

$$f(1-f) \overset{(1)}{=} \frac{1}{(e^x+1)} \frac{e^x+1-1}{e^x+1} = \frac{e^x}{(e^x+1)^2} = -\partial_x \frac{1}{e^x+1} = -\partial_x f \tag{3}$$

$$\int dE \; f(1-f) \overset{(3)}{=} \int dE \; k_B T (-\partial_E f(E)) = -k_B T \left[f(\infty) - f(-\infty) \right] = k_B T \tag{4}$$

Note: $\langle\langle Q^2 \rangle\rangle = \frac{\Delta t}{2} \big(\underbrace{4 G k_B T}_{\text{= equilibrium "Johnson-Nyquist noise"}} \big) \neq 0$, but $\propto G$ (5)