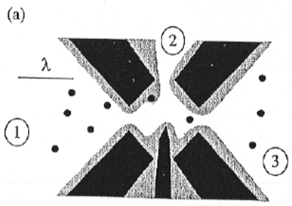
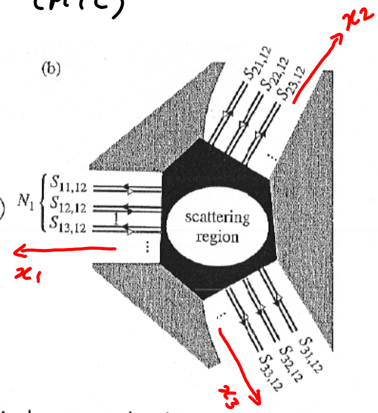
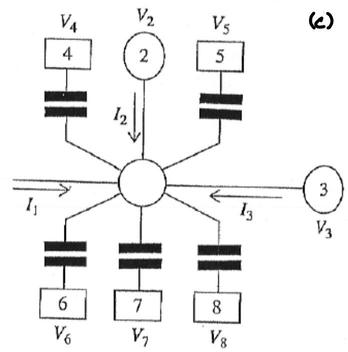
1.5 Multi-terminal circuits (MTC)

7.5.2010 MTC1







lead index: $\alpha = 1, \dots, N$
fixed voltages: V_α
coordinates: $x_\alpha, y_\alpha, z_\alpha$
transport channels: n

Multi-terminal circuit. (a) There are eight electrodes connected to this nanostructure: five gates and three terminals. (b) Structure of multi-terminal scattering matrix. We show amplitudes of the wave coming in the second channel of the first terminal.

Circuit diagram

$$\Psi(x_R, y_R, z_R) \overset{(SM2.2)}{=} \sum_m \frac{1}{\sqrt{2\pi\hbar v_m}} \Phi_m(y_R, z_R)\left[a_{Rm} e^{-ik_x^{(m)} x_R} + b_R e^{+ik_x^{(m)} x_R}\right] \qquad (1)$$

generalize.

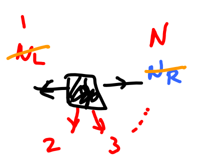
$$\Psi(x_\alpha, y_\alpha, z_\alpha) = \sum_n \frac{1}{\sqrt{2\pi\hbar v_{n\alpha}}} \Phi_{\alpha n}(y_\alpha, z_\alpha)\left[\underbrace{a_{\alpha n} e^{-ik_x^{(\alpha n)} x_\alpha}}_{\text{incoming}} + \underbrace{b_{\alpha n} e^{+ik_x^{(\alpha n)} x_\alpha}}_{\text{outgoing}}\right] \qquad (2)$$

$$k_x^{(\alpha n)} = \sqrt{2m(E - E_{\alpha n})/\hbar} \qquad (3)$$

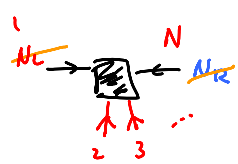
MTC2

$$b_{\alpha n} \overset{(SM3.1)}{=} \sum_{\beta = \cancel{L,R}\ 1}^{N} \sum_{m=1}^{N_\beta} S_{\alpha n, \beta m}\, a_{\beta m} \qquad \alpha = \cancel{L,R}\ 1, \dots, N \qquad m = 1, \dots, N_\beta \qquad (1)$$

↑ outgoing



↑ incoming



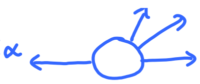
Unitarity: $\hat{S}^\dagger \hat{S} = \hat{1}, \qquad \hat{S}\hat{S}^\dagger = \hat{1}$ (2)

$\hat{S}_{\alpha\alpha}$: reflection in lead α (3)

$\hat{S}_{\alpha\beta}$: transmission from lead $\beta \to$ lead α (4)

Time reversibility: $\hat{S}(B) \overset{(SM10.2)}{=} \hat{S}^T(-B) \xrightarrow{\text{generalize}} S_{\alpha n, \beta m}(B) = S_{\beta m, \alpha n}(-B)$ (5)

Multi-terminal Landauer formula



MTC 3

Current in lead α, at $x_\alpha = \infty$:

$$I_\alpha \overset{(SM11.1)}{=} 2_s e \sum_n \int_{-\infty}^{\infty} \frac{dk_x^{(\alpha n)}}{2\pi} v_x(k_x^{(\alpha n)}) f(k_x^{(\alpha n)}) \quad (1)$$

	came from	with probability	$f_n(k_x)$	
$k_x^{(\alpha n)} < 0$	α, n	1	$f_\alpha(E)$	(2)
$k_x^{(\alpha n)} > 0$	β, m	$\lvert S_{\alpha n, \alpha' n'}\rvert^2$	$f_{\alpha'}(E)$	(3)

$$I_\alpha \overset{(SM12.1)}{=} 2_s e \sum_n \int_0^\infty \frac{dk_x}{2\pi} v_x(k_x) \left[- f_\alpha(E) + \sum_{\beta m} \lvert S_{\alpha n, \beta m}\rvert^2 f_\beta(E) \right] \quad (4)$$

$$= - \frac{1}{e} G_Q \sum_n \int dE \sum_\beta \left[\delta_{\alpha\beta} \delta_{nm} - \lvert S_{\alpha n, \beta m}\rvert^2 \right] f_\beta(E) \quad (5)$$

$$\boxed{I_\alpha = - \frac{1}{e} G_Q \int dE \sum_\beta \mathrm{Tr}\left\{ \delta_{\alpha\beta} - \hat{S}^\dagger_{\beta\alpha} S_{\alpha\beta} \right\} f_\beta(E)} \quad (6)$$

over channels

Current conservation: (total current in all terminals vanishes)

MTC 4

$$\sum_\alpha I_\alpha \overset{(3.6)}{=} - \frac{G_Q}{e} \int dE \sum_\beta \mathrm{Tr}\Big\{ 1 - \underbrace{(\hat{S}^\dagger \hat{S})_{\beta\beta}}_{=1} \Big\} f_{\alpha'}(E) = 0 \quad ✓ \quad (1)$$

unitarity:

Linear regime: apply small voltage only to one lead, say γ

$$\mu_\gamma = \varepsilon_F + eV_\gamma , \qquad \mu_\beta = \varepsilon_F \quad \forall \beta \neq \gamma \quad (2)$$

$$f_\beta(E) = \left[e^{\beta(E - \mu_\beta)} + 1 \right]^{-1}$$

gives 0, since $\mathrm{Tr}\left(1 - (\hat{S}\hat{S}^\dagger)_{\alpha\alpha}\right) = 0$

$$I_\alpha = - \frac{1}{e} G_Q \int dE \sum_\beta \mathrm{Tr}\left\{ \delta_{\alpha\beta} - \hat{S}^\dagger_{\beta\alpha} \hat{S}_{\alpha\beta} \right\} \underbrace{\left(f_\beta(E) - f(E) \right)}_{\neq 0 \text{ only for } \beta = \gamma} \quad (3)$$

if $\hat{S}_{\alpha\beta}$ is E-independent on scale of eV:

$$\boxed{I_\alpha = \underbrace{- G_Q \mathrm{Tr}\left[\delta_{\alpha\gamma} - \hat{S}^\dagger_{\gamma\alpha} \hat{S}_{\alpha\gamma} \right]}_{\equiv G_{\alpha\gamma} \; (5)} V_\gamma = G_{\alpha\gamma} V_\gamma} \quad \text{"multi-terminal Landauer formula"} \quad (4)$$

In linear regime, contributions to I_α from all leads with $V_\beta \neq 0$ add up: MTC5

$$I_\alpha = \sum_\beta G_{\alpha\beta} V_\beta \qquad [\, G_{\alpha\beta} : \text{elements of "conductance matrix} \,] \qquad (1)$$

Current conservation: $0 \overset{(4.1)}{=} \sum_\alpha I_\alpha$ for arbitrary $V_\beta \Rightarrow \boxed{\sum_\alpha G_{\alpha\beta} = 0 \quad \forall \beta}$ (2)

Time reversibility (2.5) implies:
$S_{\alpha n, \beta m}(B) = S_{\beta m, \alpha n}(-B)$

$$\boxed{G_{\alpha\beta}(B) = G_{\beta\alpha}(-B)} \quad \text{"Onsager relations"} \qquad (3)$$

For $B = 0$, $G_{\alpha\beta}$ is symmetric; # of independent parameters then:

(2) $\Rightarrow$ only $(N-1)$ indep. parameters per column.

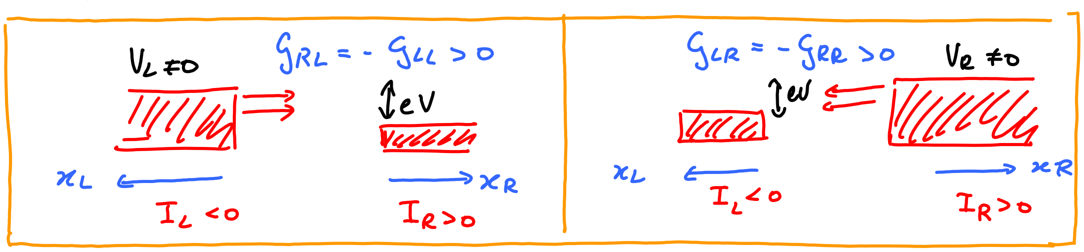
$(N-1)$ diagonal elements, and $[(N-1)^2 - (N-1)]/2$ off-diag. are independent. since G is symmetric

In total: $[(N-1)^2 + (N-1)]/2 = N(N-1)/2$ (3)

For $N = 2$: only 1 independent element: MTC6

$$G = G_{LR} = G_{RL} = -G_{LL} = -G_{RR}$$

Check Signs:



Resistance matrix $R_{\alpha\beta}$:

defined by: $V_\alpha = \sum_\beta R_{\alpha\beta} I_\beta \overset{(5.1)}{\Longrightarrow} R = G^{-1}$ (1)

$$R_{\alpha\beta}(B) = R_{\beta\alpha}(-B) \qquad \text{"Onsager relations"} \qquad (2)$$

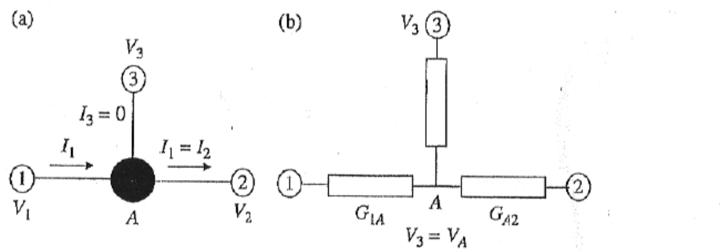
Voltage probes and two-terminal measurements (R. Landauer)

MTC7

Ideal voltage probe:
non-invasive, does not
draw current!

$$I_\alpha \overset{(5.1)}{=} \sum_\beta g_{\alpha\beta} V_\beta$$

$$0 \overset{(5.2)}{=} \sum_\beta g_{\alpha\beta}$$



(a) Three-terminal circuit with terminal 3 as a voltage probe. (b) A classical circuit could be presented in this way, and conductances of the elements G_{1A}, G_{A2} can be determined from the voltage V_3 measured.

$$0 = I_3 = g_{31} V_1 + g_{32} V_2 + \underbrace{g_{33}}_{\overset{(5.2)}{=} -[g_{31} + g_{32}]} V_3 = g_{31}(V_1 - V_3) + g_{32}(V_2 - V_3) \tag{1}$$

↑ for ideal voltmeter

Solve: $$V_3 = \frac{g_{31} V_1 + g_{32} V_2}{g_{31} + g_{32}} \overset{(4.5)}{=} \frac{V_1 \, \mathrm{Tr}(\hat{S}_{13}^{\dagger} \hat{S}_{31}) + V_2 \, \mathrm{Tr}(\hat{S}_{23}^{\dagger} \hat{S}_{32})}{\mathrm{Tr}(\hat{S}_{13}^{\dagger} \hat{S}_{31}) + \mathrm{Tr}(\hat{S}_{23}^{\dagger} \hat{S}_{32})} \tag{2}$$

"Noninvasive measurement" requires: $g_{31}, g_{32} \ll g_{12}$, or $\hat{S}_{31}, \hat{S}_{32} \to 0$. (3)

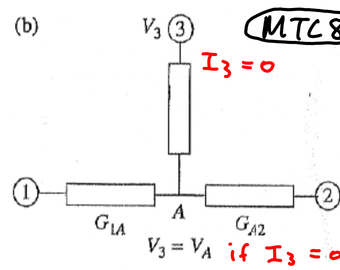
Then V_3 remains finite!

MTC8

Finite V_3 is also obtained in classical circuit model:

Consider g_{1A} and g_{A2} in series, $V_3 = V_A$, then

$$\frac{g_{1A}}{g_{12}} = \frac{V_1 - V_2}{V_1 - V_A} \quad (1a) \qquad \frac{g_{A2}}{g_{12}} = \frac{V_2 - V_1}{V_2 - V_A} \quad (1b)$$



$\Rightarrow$ g_{1A}, g_{1B} can be measured by measuring V_1, V_2, V_A.

Consistency checks: Solve for V_A:

compare with (7.2)

$$(V_1 - V_A) g_{1A} = g_{12}(V_1 - V_2) \Rightarrow V_A = \frac{1}{g_{1A}}\left[V_1 (g_{1A} - g_{12}) + g_{12} V_2\right] \tag{2a}$$

$$(V_2 - V_A) g_{A2} = g_{12}(V_2 - V_1) \Rightarrow V_A = \frac{1}{g_{A2}}\left[V_2 (g_{A2} - g_{12}) + g_{12} V_1\right] \tag{2b}$$

$\neq V_3$ of (7.2) !!

$$(2a) = (2b) \Rightarrow R_{1A}\left[V_1\left(\frac{1}{R_{1A}} - \frac{1}{R_{12}}\right) + \frac{V_2}{R_{12}}\right] = R_{A2}\left[V_2\left(\frac{1}{R_{A2}} - \frac{1}{R_{12}}\right) + \frac{1}{R_{12}}\right] \tag{3}$$

$$\Rightarrow V_1\left[1 - \frac{R_{1A} + R_{A2}}{R_{12}}\right] = V_2\left[1 - \frac{R_{A2} + R_{A1}}{R_{12}}\right] \tag{4}$$

(4) must hold for any V_1, V_2 $\Rightarrow$ $R_{1A} + R_{A2} = R_{12}$ ✓

Difference in descriptions: MTC9

Classical: we can apply a voltage to any point in circuit, including A

Quantum: we can apply voltage only to reservoirs; if A represents a nanostructure (e.g. single-channel conductor), it is too small to apply a voltage directly to it.

Lets see what happens if we try to assign voltages "inside a nanostructure":

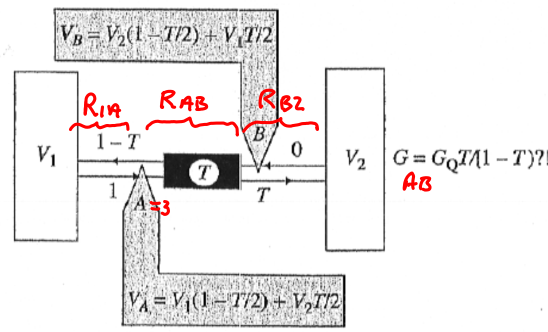
If we write $R_{tot} = R_{1A} + R_{AB} + R_{B2}$, what is R_{AB}?

In this classical picture, we would have (in analogy to (8.1)):

$$g_{1A} = g_{12} \frac{V_1 - V_2}{V_1 - V_A} \qquad (1)$$

$$g_{AB} = g_{12} \frac{V_1 - V_2}{V_A - V_B} \qquad (2)$$

$$g_{B2} = g_{12} \frac{V_1 - V_2}{V_B - V_2} \qquad (3)$$



So: First determine V_A, V_B, then we get g_{AB} from (2).

Let us get $V_A = V_3$ from (7.2): MTC10

w: probability to enter A

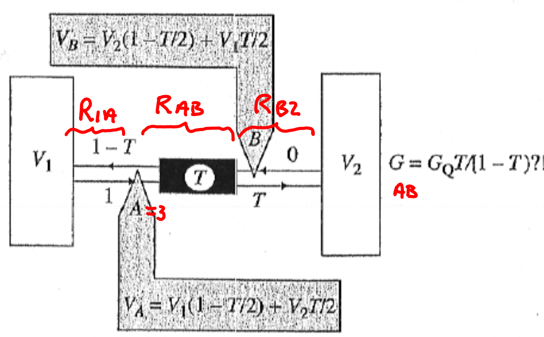
$$V_A = V_3 \overset{(7.2)}{=} \frac{g_{31} V_1 + g_{32} V_2}{g_{31} + g_{32}} \qquad (1)$$

We first need:

$$g_{31} = g_{A1} = g_Q \left[w + (1-T)(1-w)w\right] \simeq$$

$$\simeq g_Q\, w(2-T) \qquad (2a)$$

$$g_{32} = g_{A2} = g_Q\, w\, T \qquad (2b)$$



(2) into (1):
$$V_A = \frac{(2-T)V_1 + T V_2}{(2-T) + T} = (1 - T/2)V_1 + T/2\, V_2 \overset{!!}{\neq} V_1 \qquad (3a)$$

Similarly:
$$V_B = (1 - T/2)V_2 + T/2\, V_1 \overset{!!}{\neq} V_2 \qquad (3b)$$

That $V_A \neq V_1$, $V_B \neq V_2$ is a consequence of pretending that voltage V_A can be applied directly inside nanostructure ...

Let us proceed nevertheless and insert V_A, V_B from (10.2) into (9.1-3): MTC 9

$$1\to A: \quad g_{1A} \overset{(9.1)}{=} g_{12}\frac{V_1-V_2}{V_1-V_A} \overset{(8.3a)}{=} g_Q \cancel{T}\frac{V_1-V_2}{\cancel{V_1}-[\cancel{(1}-\cancel{T}/2)V_1+\cancel{T}/2\,V_2]} = 2g_Q \quad (1)$$

$$A\to B: \quad g_{AB} \overset{(9.2)}{=} g_{12}\frac{V_1-V_2}{V_A-V_B} \overset{(8.3)}{=} g_Q T\frac{V_1-V_2}{[(1-T/2)V_1+T/2\,V_2]-[(1-T/2)V_2+T/2\,V_1]}$$

$$= \frac{g_Q T}{1-T} \quad \left(\neq g_Q T = \text{two-terminal Landauer formula !!!}\right) \quad (2)$$

$$B\to 2: \quad g_{B2} \overset{(9.3)}{=} g_{12}\frac{V_1-V_2}{V_B-V_2} = g_Q\cancel{T}\frac{V_1-V_2}{[\cancel{(1}-\cancel{T}/2)V_2+\cancel{T}/2\,V_1]-\cancel{V_2}} = 2g_Q \quad (3)$$

Consistency check:

$$R_{tot} = R_{1A}+R_{AB}+R_{B2} = \frac{1}{g_{1A}}+\frac{1}{g_{AB}}+\frac{1}{g_{B2}} = \frac{1}{g_Q}\left[\frac{1}{2}+\frac{1-T}{T}+\frac{1}{2}\right] = \frac{1}{g_Q T} = \frac{1}{g_{tot}} \checkmark \quad (4)$$

Comments: MTC 10

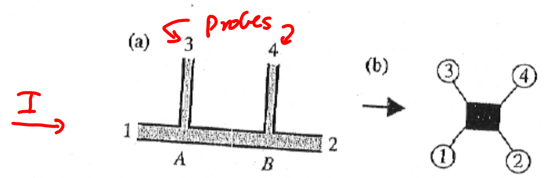
- $$g_{AB} = \underbrace{\frac{g_Q T}{1-T}}_{=\text{ "wrong Landauer formula"}} \neq g_Q T \quad \text{as expected from Landauer formula}$$
- For QPC with $T=1$: $g_{AB}\to\infty$ or $R_{AB} = g_{AB}^{-1}\to 0$

  this does make sense: QPC has zero resistance and no voltage drop across it !
- Voltage drops across $R_{1A} = \frac{1}{g_{1A}} = \frac{1}{2g_Q}$ and $R_{1B} = \frac{1}{g_{1B}} = \frac{1}{2g_Q}$

  (called "contact resistances" by Landauer).
- What went wrong: in reality, it is not possible to apply voltages V_A and V_B to a single-channel scatterer. Voltages can be applied only to reservoirs. It is not possible to separate nanostructure into separate elements of definite resistance.
- Tunnel junction: $T \ll 1$: $g_{AB} \simeq g_Q T = g_{tot}$ ✓ as expected.

Reason: voltage drops across tunnel junction, contribution of contact resistances is $\simeq 0$

If Eqs. (11.1) - (11.3) are to hold, we need a 4-terminal circuit: MTC 13



A common scheme for two-probe measurement of the resistance R_{AB}. (a) does not generally work in quantum transport. The reason is that the equivalent four-terminal nanostructure (b) cannot be generally separated into elements of definite resistance.

Classically: terminals 1 & 2 pass current

terminals 3 & 4 probe voltage drop between A & B

we expect $R_{AB} = \dfrac{V_3 - V_4}{I}$

Quantum transport: this does not work in general!

Eg: assume non-invasive voltage probes: $g_{43}, g_{34} \ll g_{41}, g_{42}$

$g_{31}, g_{32} \ll g_{12}, g_{21}$

MTC 14

Using $I_\alpha \overset{(5.1)}{=} \sum_\beta g_{\alpha\beta} V_\beta$, $0 \overset{(5.2)}{=} \sum_\beta g_{\alpha\beta}$, (1)

we get from
$$0 = I_3 = g_{31} V_1 + g_{32} V_2 + \underbrace{g_{33}}_{= -(g_{31} + g_{32})} V_3$$
$$0 = I_4 = g_{41} V_1 + g_{42} V_2 + \underbrace{g_{44}}_{= -(g_{41} + g_{42})} V_4$$

$$V_3 - V_4 = (V_1 - V_2)\left(\frac{g_{31}}{g_{31} + g_{32}} - \frac{g_{41}}{g_{41} + g_{42}}\right) \qquad (2)$$

Current: $I = g_{21}(V_1 - V_2)$ (3)

$$R = \frac{V_3 - V_4}{I} = \frac{1}{g_{21}}\left(\frac{g_{31}}{g_{31} + g_{32}} - \frac{g_{41}}{g_{41} + g_{42}}\right) \neq R_{AB} \qquad (4)$$

Note also $R(B) \neq R(-B)$: Onsager relations do not hold in multiterminal setup.