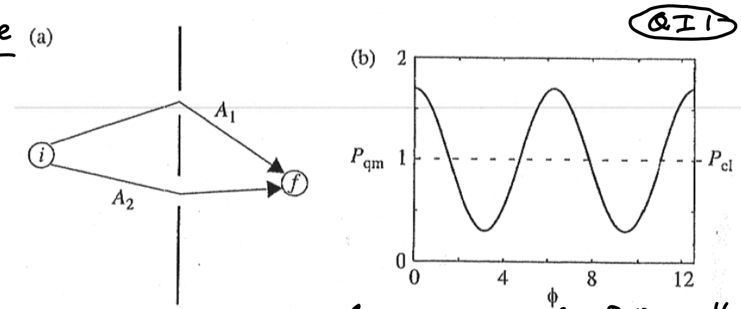
1.6 Quantum Interference

QI 1

Double-slit experiment:



Interference pattern for $P_1/P_2 = 1/6$

Probability for trajectory i: $\quad P_i = |A_i|^2, \quad i = 1, 2$ (1)

Total probability, classical: $\quad P_{cl} = P_1 + P_2$ (2)

quantum mechanical: $\quad P_{qm} = |A_1 + A_2|^2 = P_1 + P_2 + \underbrace{2\mathrm{Re}\, A_1 A_2^*}_{\text{interference term}}$ (3)

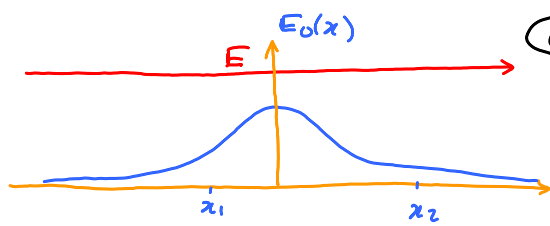
write $A_i = \sqrt{P_i}\, e^{i\phi_i}$, $\phi = \phi_1 - \phi_2$: $\quad = P_{cl} + 2\sqrt{P_1 P_2}\cos\phi$ (4)

$P_{qm} \gtrless P_{cl}$: $\left\{\begin{matrix}\text{constructive}\\ \text{destructive}\end{matrix}\right\}$ interference. For $P_1 = P_2$, $\phi = \pi \Rightarrow P_{qm} = 0$ (5)

1.6.1 Phase shifts

QI 2

1D motion, smooth potential, no scattering.



Semi-classical Ansatz for wave function (WKB):

$$\psi(x) \propto e^{i\phi(x)}, \qquad \frac{d\phi}{dx} = k(x) \equiv \sqrt{2m(E - E_0(x))}/\hbar \quad (1)$$

[Check: $\frac{\hbar^2 \partial_x^2}{2m}\psi(x) = \frac{\hbar^2}{2m} 2m(E - E_0)/\hbar^2 \; \psi(x) = (E - E_0(x))\psi(x)$

[+ terms in $\frac{\partial E_0(x)}{\partial x}$ which are assumed small for "smooth potential"]]

Phase difference between x_1 and x_2: $\quad \phi = \phi(x_2) - \phi(x_1) \overset{(1)}{=} \int_{x_1}^{x_2} dx\, k(x)$ (2)

If potential $E_0(x)$ = konst.: $\quad \phi = Lk, \quad L = x_2 - x_1$ (3)

Absolute value of phase is usually not accessible. Study how it changes!   QI3

E-dependence:

$$\frac{d\phi}{dE} \overset{(2.2)}{=} \int_{x_1}^{x_2} dx \, \underbrace{\frac{d k(x)}{dE}}_{= 1/\hbar v(x)} = \int_{x_1}^{x_2} dx \, \frac{1}{\hbar} \frac{dt}{dx} = \frac{1}{\hbar}\int_{t_1}^{t_2} dt = \frac{\tau_{21}}{\hbar} \tag{1}$$

$\tau_{21} = \tau_2 - \tau_1$ = time of flight

Modify potential: $E_0(x) \to E_0(x) + eV(x)$ (small compared to E_0) $k(x) = \sqrt{2m(E - E_0(x))}$

$$\Delta\phi = \int_{x_1}^{x_2} dx \left[k^{V(x)\neq 0}(x) - k^{V(x)=0}(x) \right] = \int_{x_1}^{x_2} dx \, \frac{dk(x)}{dE}\left[-eV(x)\right] \tag{2}$$

$$= -\frac{1}{\hbar}\int_{x_1}^{x_2} dx \, \frac{dt}{dx} \, eV(x) = -\frac{1}{\hbar}\int_{t_1}^{t_2} dt \, eV(x(t)) = -\frac{\tau}{\hbar} eV \tag{3}$$

trajectory

[if $V(x) = V = \text{const.}$]

This is consistent with what we know for 1D - Schrödinger eq:

QI4

if $\left[-\frac{\hbar^2}{2m}\partial_x^2 - i\hbar\partial_t \right] \psi_0(x,t) = 0$, (1)

then $\left[-\frac{\hbar^2}{2m}\partial_x^2 + V(t) - i\hbar\partial_t \right] \psi(x,t) = 0$ (2)

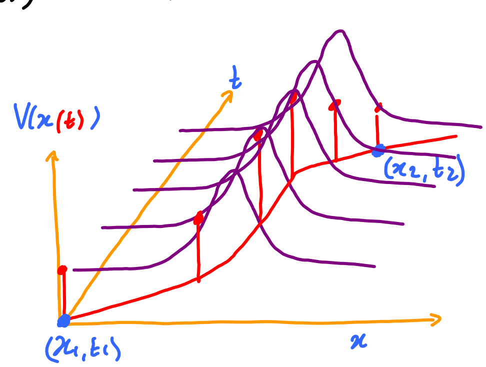
has solution $\psi(x,t) = e^{-\frac{i}{\hbar}\int^t dt' V(t')} \psi_0(x,t)$ (3)

Analogously: Phase shift for motion in 3D along classical trajectory $\vec{x}(t)$:

$$\frac{d\phi}{dE} = \frac{1}{\hbar}\int_{t_1}^{t_2} dt \overset{(1)}{=} \frac{\tau}{\hbar} \; ; \quad \Delta\phi \overset{(3)}{=} \int_{t_1}^{t_2} \frac{dt}{\hbar} \, eV(\vec{x}(t)) \tag{4}$$

"Dynamical phase" due to energy or potential:
depends only on duration of trajectory,
not on direction.

$$\Delta\phi\,(x_1 \to x_2) = \Delta\phi\,(x_2 \to x_1) \tag{5}$$



Phase shift in homogeneous magnetic field : $\vec{B} = \vec{\nabla} \times \vec{A}$:

QI5

$$\phi_{mag} = \frac{e}{\hbar c} \int_{\vec{x}_1}^{\vec{x}_2} d\vec{x}' \cdot \vec{A}(\vec{x}') = \frac{e}{\hbar c} \int_{t_1}^{t_2} dt \; \vec{A}(\vec{x}(t)) \cdot \vec{v} \quad (1)$$

Aharonov-Bohm phase: $\phi_{mag}(x_1 \to x_2) = - \phi_{mag}(x_2 \to x_1)$ (2)

Why (1) ? Recall: For free particle in magnetic field, Hamiltonian is :

$$\text{Let} \quad \left[\frac{1}{2m} (-\hbar^2 \vec{\nabla})^2 - i\hbar \partial_t \right] \psi_0(\vec{x}, t) = 0 \quad (2)$$

In magnetic field $\vec{p} \to \vec{p} - e/c \vec{A}$ (minimal coupling) (3)

$$\text{then} \quad \left[\frac{1}{2m} (-i\hbar \vec{\nabla} - e/c \vec{A})^2 - i\hbar \partial_t \right] \psi(\vec{x}, t) = 0 \quad (4)$$

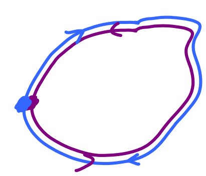
$$\text{has solution} \quad \psi(\vec{x}, t) = e^{i\phi_{mag}(\vec{x})} \psi_0(\vec{x}, t) \quad (5)$$

$$\text{provided that} \quad -i\hbar \vec{\nabla} \, i\phi_{mag}(\vec{x}) = e/c \vec{A} \quad \Rightarrow (1) \quad (6)$$

QI6

ϕ_{mag} is not gauge-invariant (it changes under $\vec{A} \to \vec{A} + \vec{\nabla}\chi$), and hence "unphysical" or unobservable.

Phase accumulated along closed path is observable :



$$\phi_{mag} = \frac{e}{\hbar c} \oint \vec{A} \cdot d\vec{x} \overset{\text{Stokes}}{=} \frac{e}{\hbar c} \underbrace{\oint \underbrace{(\vec{\nabla} \times \vec{A})}_{\vec{B}} \cdot d\vec{S}}_{\Phi} \quad (1)$$

$$= \frac{\pi \Phi}{\Phi_0} \quad \text{with flux quantum} \quad \Phi_0 = \frac{\pi \hbar c}{e} = \frac{hc}{2e} \quad (2)$$

(gauge-invariant!)

[some authors omit 2 here]

Phase difference between forward and backward trajectories :

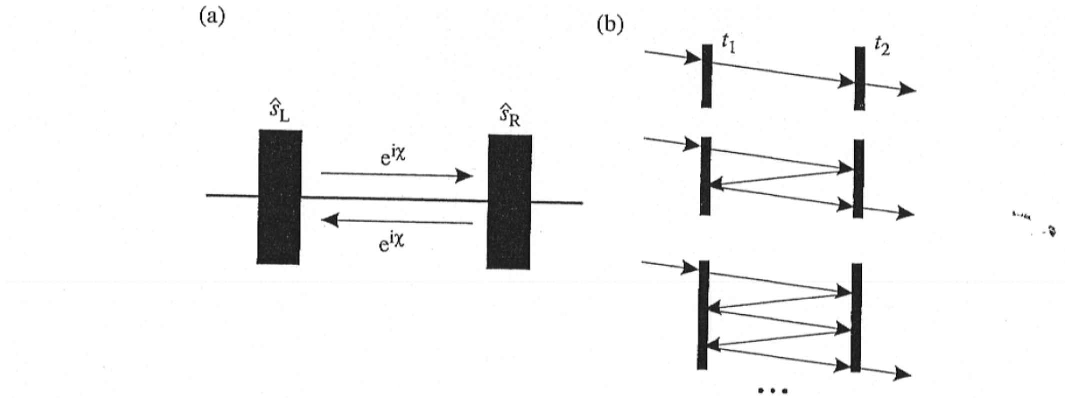
$$\boxed{\phi^{\circlearrowleft}_{mag} - \phi^{\circlearrowright}_{mag} = 2\pi \, \Phi / \Phi_0} \quad (3)$$

Some physical quantities depend periodically on $\Phi/\Phi_0 \Rightarrow$ Aharonov-Bohm effect.

For double slit, p. QI1: $\phi_1 - \phi_2 = \frac{\pi \Phi}{\Phi_0}$ (4) $\Rightarrow$ interference pattern shifts if Φ is changed.

1.6.2 Double Junction

QI7



(a) Double-junction nanostructure consists of two scatterers in series. The electron acquires the phase shift χ when traveling between the scatterers. (b) The transmission results from the interference of all trajectories shown, and thus depends on the phase shift.

Left junction:

$$\hat{S}_L = \begin{pmatrix} \hat{r}_L & \hat{t}'_L \\ \hat{t}_L & \hat{r}'_L \end{pmatrix}$$

Between junctions:

Phase shift

$$e^{i\chi}$$

Right junction:

$$\hat{S}_R = \begin{pmatrix} \hat{r}_R & \hat{t}'_R \\ \hat{t}_R & \hat{r}'_R \end{pmatrix}$$

Electron can get from L to R in many ways ("processes"), differing in number of attempts (m) to cross right junction. Amplitudes for processes add coherently!

QI8

amplitude for 1 attempt: $A_1 = t_L e^{i\chi} t_R$ (1)

2 attempts: $A_2 = t_L e^{i\chi} r_R e^{i\chi} r'_L e^{i\chi} t_R$ (2)

m attempts: $A_m = t_L t_R e^{i\chi} (r_R r'_L)^{m-1} e^{i2(m-1)}$ (3)

total amplitude: $t = \sum_{m=1}^{\infty} A_m = t_L t_R e^{i\chi} \sum_{m=0}^{\infty} \left(r_R r'_L e^{2i\chi}\right)^m$ (4)

$\left[T_i = |t_i|^2 = 1 - R_i \quad i = L, R\right]$

$$= \frac{t_L t_R e^{i\chi}}{1 - \underbrace{r_R r'_L e^{2i\chi}}_{\equiv \sqrt{R_R R_L}\, e^{2i\chi'}}} \quad (5)$$

total phase accumulated

transmission probability: $T = |t|^2 = \dfrac{T_L T_R}{1 + R_R R'_L - 2\sqrt{R_R R_L} \cos 2\chi'}$ (6)

Transmission depends periodically on $\chi \overset{(3.1)}{=} \underbrace{E\,\tau_{12}}_{\text{roundtrip time}}/\hbar$ (1) QI9

$$T_{\substack{max\\min}} = \frac{T_L T_R}{\left[1 \mp \sqrt{R_L R_R}\right]^2}\ , \quad \text{achieved for } \chi = \begin{cases} 0 \\ \pi \end{cases} \tag{2}$$

Consider limit of small transparency for both junctions: $T_L, T_R \ll 1$. (3)

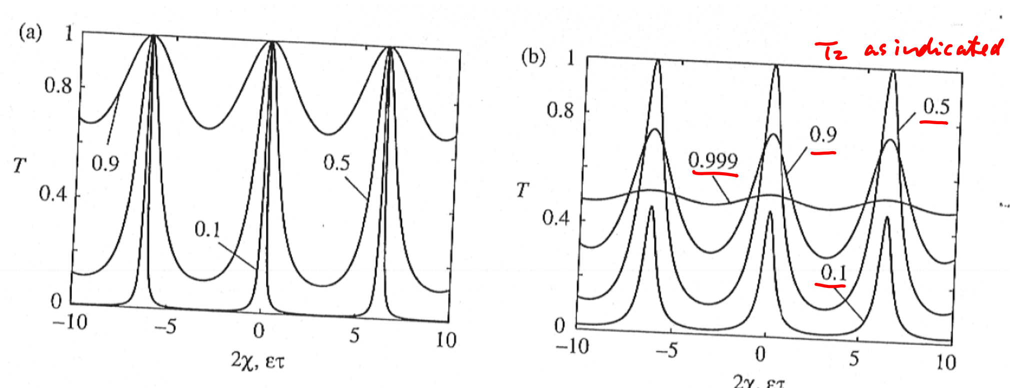
$$T_{min} \simeq T_L T_R$$

$$T_{max} = \frac{T_L T_R}{\left[1 - \underbrace{(1-T_L)^{1/2}(1-T_R)^{1/2}}_{\simeq\,(1 - \frac{1}{2}T_L - \frac{1}{2}T_R \,\ldots)}\right]^2} \simeq \frac{4\,T_L T_R}{(T_L + T_R)^2} = \begin{cases} 1 \quad \text{for} \\ T_L = T_R\ ! \end{cases} \tag{4}$$

T(E) has resonant structure, with peaks when $\chi = \pi n$ (5)

"resonant tunneling" or "Fabry-Perot resonances".
Resonances indicate formation of discrete levels,
essentially isolated from outside world. (6)

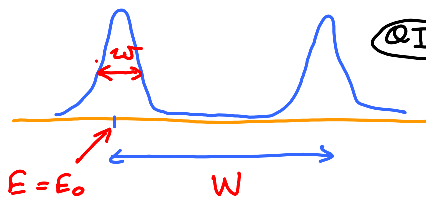
symmetric scatterers, $T_1 = T_2$ — non-symmetric: $T_1 = 0.5$ QI10



Transmission via double junction versus phase shift or energy. (a) Symmetric scatterers with $T_1 = T_2$ corresponding to the curve labels. The maximum transmission is always 1 in this case. (b) Non-symmetric scatterers, $T_1 = 0.5$, and T_2 corresponds to the curve labels. The narrow transmission resonances for $T_{1,2} \ll 1$ indicate formation of discrete energy levels at corresponding energies.

Expand T(E) around resonance:

QI 11



$$\chi' = (E - E_0)/W \qquad (1)$$

$$\cos\chi' = 1 - \tfrac{1}{2}\chi'^2 \qquad (2)$$

$$T(E) \overset{(8.6)}{=} \frac{T_L T_R}{\underbrace{1 + R_R R_L - 2\sqrt{R_R R_L'}}_{T_L, T_R \ll 1:\ 9.4\ \simeq\ (T_L + T_R)^2/4}\left(1 - \tfrac{1}{2}\chi'^2\right)} = \frac{T_L T_R}{\left(\frac{T_L + T_R}{2}\right)^2 + \left(\frac{E - E_0}{W}\right)^2} \qquad (3)$$

T(E) = Lorentzian of full width at half maximum : $w = W(T_L + T_R)$ (4)

$$\frac{w}{\hbar} = (\text{decay rate}) = (\text{probability per unit time to escape to } L \text{ or } R) \qquad (5)$$

$$= (\text{attempt frequency}) \times (\text{success prop. on } L \text{ or } R) = \left(\frac{W}{\hbar}\right)(T_L + T_R) \equiv \Gamma_L + \Gamma_R$$

$\Gamma_i \equiv \frac{W}{\hbar} T_i$ = decay rates on L/R

$$T(E) \overset{T_i = (\hbar/W)\Gamma_i}{=} \frac{\Gamma_L \Gamma_R}{\left(\frac{\Gamma_L + \Gamma_R}{2}\right)^2 + \left(\frac{E - E_0}{\hbar}\right)^2} \qquad (6)$$

"Breit-Wigner formula"

Classical probability to get to right:

QI 12

$$T_{cl} = \sum_{m=0}^{\infty} |A_m|^2 \overset{(8.3)}{=} \underbrace{|t_L t_R|^2}_{T_L T_R} \sum_{m=0}^{\infty} \underbrace{|r_L' r_R|}_{R_L R_R}{}^{2m} = \frac{T_L T_R}{\underbrace{1 - R_L R_R}_{T_L + T_R - T_L T_R}}$$

= independent of energy or phase . Thus resonant tunneling is pure quantum effect!

$$g_{cl} = g_Q T_{cl} \overset{T_L, T_R \ll 1}{=} \frac{g_Q T_L\, g_Q T_R}{g_Q (T_L + T_R)} = \frac{g_L g_R}{g_L + g_R} = \frac{1}{g_R} + \frac{1}{g_L}$$

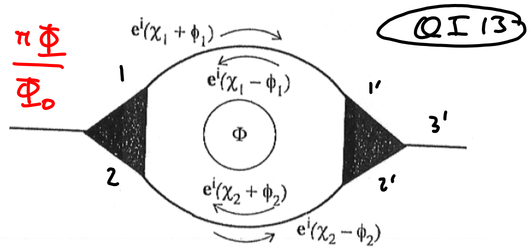
= Ohm's law for resistors in series.

Any effects beyond Ohm's law results from quantum interference...!

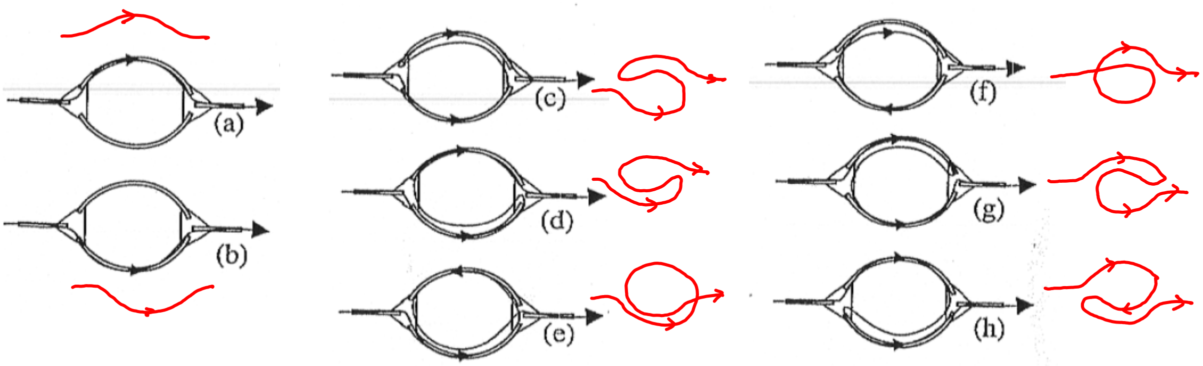
1.6.3 Aharonov-Bohm (AB) ring

$\phi_1 + \phi_2 = \frac{\pi \Phi}{\Phi_0}$

QI 13



$$\hat{S}_L = \begin{pmatrix} S_{11} & S_{21} & S_{31} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} = \hat{S}_R$$

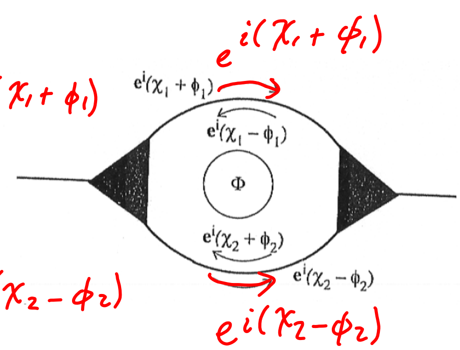


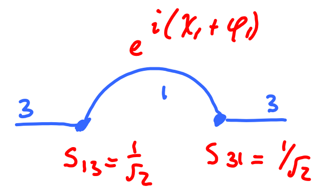
Several example trajectories interfering in an AB ring. Interference of (a) and (b) provides a Φ_0 periodic contribution that depends on dynamical phase (*universal conductance fluctuations*). Interference in pairs (d), (e) and (f), (g) leads to a Φ_0 periodic contribution that survives averaging over dynamical phase (*weak localization*). Interference in pairs (c), (h) and (g), (d) yields a contribution time-reversed to that of pair (a), (b).

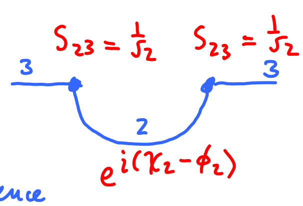
QI 14

Find amplitudes for various trajectories:

$$\hat{S}_L = \begin{pmatrix} S_{11} & S_{21} & S_{31} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} = \hat{S}_R$$



$t_{(a)} =$  $= \frac{1}{2} e^{i(\chi_1+\phi_1)}$

$t_{(b)} =$  $= \frac{1}{2} e^{i(\chi_2-\phi_2)}$

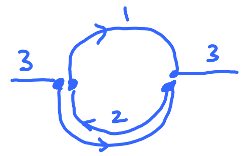
$\Phi_{AB} = \pi \Phi / \Phi_0 \Rightarrow$ AB-effect

interference

$$p^{int}_{ab} = 2 \,\mathrm{Re}\, t_{(a)} t^*_{(b)} \propto \cos(\underbrace{\chi_1 - \chi_2} + \overbrace{\phi_1 + \phi_2})$$

depends on dynamical phases! will average to zero upon ensemble averaging

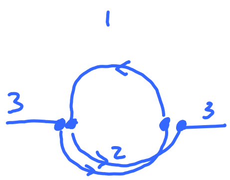
$\Rightarrow$ "universal conductance fluctuations"

$t_{(d)} =$ 

$$= \overset{\frac{1}{\sqrt{2}}}{S_{23}}\, e^{i(\cancel{\chi_2}-\cancel{\phi_2})}\, \overset{-1/2}{S_{22}}\, e^{i(\cancel{\chi_2}+\cancel{\phi_2})}$$

$$\times \underset{\frac{1}{2}}{S_{12}}\, e^{i(\cancel{\chi_1}+\phi_1)}\, \underset{\frac{1}{\sqrt{2}}}{S_{31}}$$

$$\begin{pmatrix} -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$t_{(e)} =$ 

$$= \overset{\frac{1}{\sqrt{2}}}{S_{23}}\, e^{i(\cancel{\chi_2}-\phi_2)}\, \overset{1/2}{S_{12}}\, e^{i(\cancel{\chi_1}-\phi_1)}$$

$$\underset{1/2}{S_{21}}\, e^{i(\cancel{\chi_2}-\phi_2)}\, \underset{1/\sqrt{2}}{S_{32}}$$

$$P^{int}_{de} = 2\,\mathrm{Re}\; t_{(d)}\, t^*_{(e)} \sim \cos\left(2\phi_1 + 2\phi_2\right) = \cos\left(2\Phi_{AB}\right) \qquad (1)$$

Independant of dynamical phases!
Survives ensemble averaging!

"weak localization":
occurs whenever there are closed loops that can be traversed in opposite directions!