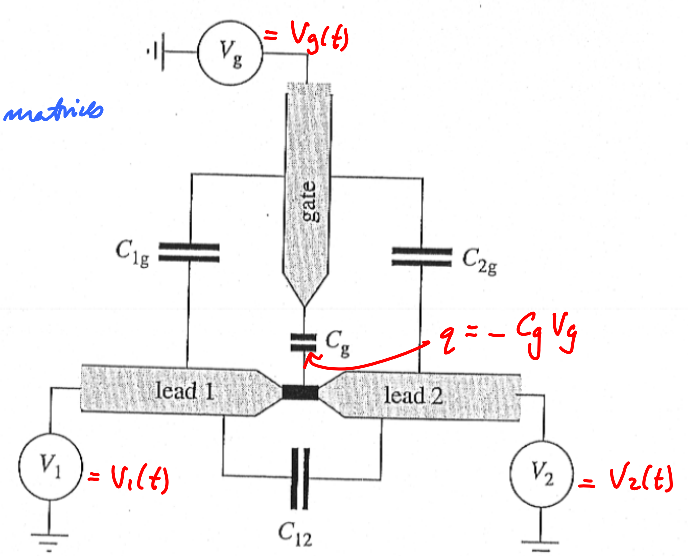
Time-dependent Transport (TDT)

TDT1

time-dependent gates

$\Rightarrow$ time-dependent scattering matrices



Nanostructure with two leads and a gate. The large external capacitances C_{12}, C_{1g}, and C_{2g} provide paths for ac current, shunting the nanostructure at sufficiently high frequencies.

TDT2

Displacement currents at finite frequencies: $I_{dis} = \dot{Q} = C\dot{V}$

Particle current: $I_{part} = gV$

$I_{dis} < I_{part} \Rightarrow C\omega V \ll gV$

$\Rightarrow \omega \ll g/C \simeq \left(\frac{1}{k\Omega}\right) \cdot 10^{10} F \simeq 10^{7}$ Hz

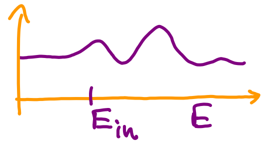
= small compared to other energy scales
($1K = 20$ GHz $= 2 \times 10^{10}$ Hz)

$\Rightarrow$ For ω small enough that displacement currents are negligible, we essentially have $\omega \simeq 0$.

Small-frequency response: TDT3

Scattering approach valid only if $\omega < E_{in}$,

where $S(E)$ becomes E-dependent for $E > E_{in}$



Typically : $E_{in} \simeq \dfrac{\hbar}{\text{traversal time}}$.

[For $\omega > E_{in}$, electrons don't traverse nanostructure, but oscillate back and forth in it.]

Charge accumulation can be ignored only for

$$\omega \lesssim \frac{1}{\tau_{RC}} = \frac{g}{C_g} \gg \frac{g}{C} = \omega_{ep}$$

(since $C_g \ll C$)

$\dot{\rho} + \vec{\nabla} \vec{j} = 0 \Rightarrow$

$J_1(\omega) - J_2(\omega) = -i\omega N(\omega)$

accumulated charge

[$\omega > \frac{1}{\tau_{RC}}$: capacitative response $\Rightarrow$ currents go via capacitors.]

$\Rightarrow$ Rather study DC current in presence of time-dependent driving / noise.

1.7.2 Tien-Gordon effect

TDT4

$$V(t) = V + \underbrace{\tilde{V} \sin \Omega t}_{\equiv V_{ac}(t)} \qquad \text{[spatially uniform]} \tag{1}$$

I-V curve in absence of modulation, $\tilde{V} = 0$

Claim: $$I_{dc}(V) = \sum_{\ell} p_{\ell} \, I(V + \hbar \Omega \ell / e) \tag{2}$$

$$p_{\ell} = J_{\ell}^{2}(e\tilde{V}/\hbar\Omega) \qquad (J_{\ell} = \text{Bessel function}) \tag{3}$$

Wave function in left lead:

for $V(t) = 0$: $$\psi(\vec{r},t) = e^{-iEt/\hbar} \, \psi_E(\vec{r}) \tag{4}$$

for $V(t) \neq 0$: $$\psi(\vec{r},t) = e^{-iEt/\hbar - ie/\hbar \int^{t} dt' \, V(t')} \, \psi_E(\vec{r}) \tag{5}$$

TDT5

Suppose $V(t)$ is periodic, e.g.

$$V(t) = V + \tilde{V} \sin \Omega t \tag{1}$$

$$\int^t dt' \, V(t') = Vt - (\tilde{V}/\Omega) \cos \Omega t \tag{2}$$

Fourier series expansion

$$e^{i(e\tilde{V}/\hbar\Omega)\cos\Omega t} = \sum_{\ell=-\infty}^{\infty} a_\ell \, e^{-i\ell\Omega t} \, , \quad a_\ell = J_\ell\left(\frac{e\tilde{V}}{\hbar\Omega}\right) \tag{3}$$

Bessel function
more complicated
if $V(t)$ is more compl.

$$\psi(\vec{r},t) \overset{(4.5)}{=} \sum_{\ell=-\infty}^{\infty} a_\ell \, e^{-i/\hbar [E + eV + \ell\hbar\Omega] t} \, \psi_E(\vec{r}) \tag{4}$$

Energy distribution:

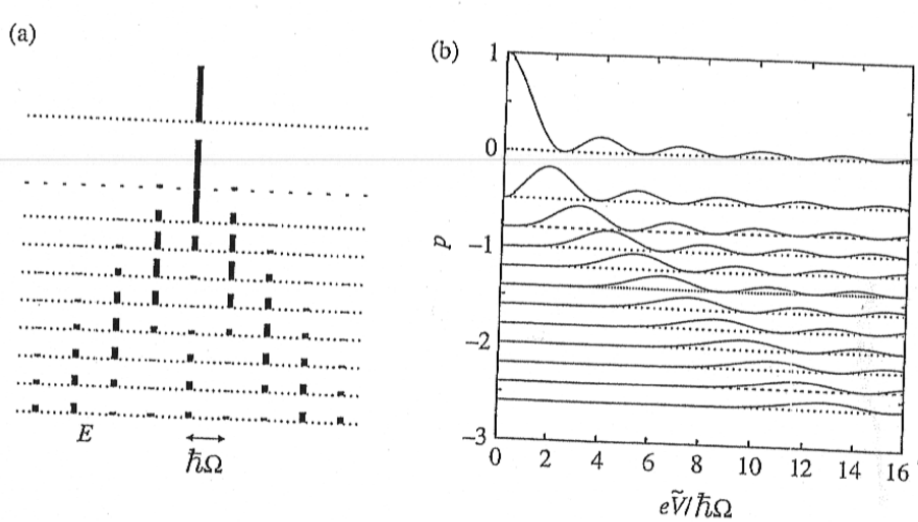
$$P_E(\varepsilon) = \sum_{\ell=-\infty}^{\infty} \underbrace{|a_\ell|^2}_{\equiv P_\ell} \, \delta(\varepsilon - E - eV - \ell\hbar\Omega) \tag{5}$$

Probability to be found in ℓ-th component.

Normalization:

$$\sum_{\ell=-\infty}^{\infty} P_\ell = 1 \tag{6}$$

TDT6



Development of side bands in the presence of ac voltage. (a) Side band weights p_l for $e\tilde{V}/\hbar\Omega$ ranging from 0 to 4.5 with step 0.5. (b) First 12 p_l plotted versus $e\tilde{V}/\hbar\Omega$; oscillations are clearly visible. The curves are offset for clarity.

TDT7

Now consider tunneling contact, with $T_p \ll 1$.

Characteristic function:

$$\boxed{\ln \Lambda_{\Delta t}(\chi) \overset{(CE9.3)}{=} 2_s\, \Delta t \int \frac{dE}{2\pi\hbar} \sum_p \ln \Big\{ 1 + T_p (e^{i\chi} - 1) f_L(E)[1 - f_R(E)] + T_p (e^{-i\chi} - 1) f_R(E)[1 - f_L(E)] \Big\}} \quad (1)$$

"Levitov formula" (1996)

$$\ln \Lambda_{\Delta t}(\chi) \overset{T_p \ll 1}{=} \Delta t \left\{ (e^{i\chi} - 1)\Gamma_{LR} + (e^{-i\chi} - 1)\Gamma_{RL} \right\} \quad (2)$$

$$\Gamma_{LR} = 2_s \int \frac{dE}{2\pi\hbar} \sum_p T_p(E)\, f_L(E)[1 - f_R(E)]$$

$$\Gamma_{RL} = 2_s \int \frac{dE}{2\pi\hbar} \sum_p T_p(E)\, f_R(E)[1 - f_L(E)] \quad (3)$$

E-dependence measured relative to ε_F in R lead $\Rightarrow$ (4)

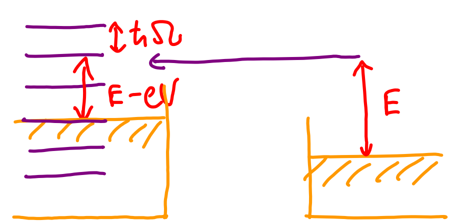
assume shaking μ_L of left lead does not modify $T_p(E)$

TDT8

Assume: for $\tilde{V} = 0$: $f_R(E) = f_F(E)$, $f_L(E) = f_F(E - eV)$ (1)

for $\tilde{V} \neq 0$: $f_R(E) \to f_F(E)$, $f_L(E) \to \sum_\ell P_\ell\, f_F(E - eV - \hbar\Omega\ell) \equiv \tilde{f}_L(E)$ (2)

Electron moving from R to L can absorb ($\ell < 0$) or emit ($\ell > 0$) ℓ quanta of energy $\hbar\Omega$, coming from AC field.



Tien-Gordon = photon-assisted tunneling.

Current: $$I_{dc} = \frac{e\langle Q \rangle}{\Delta t} \overset{(CE15.1)}{=} \frac{e}{\Delta t} \frac{\partial \ln \Lambda}{\partial(i\chi)}\bigg|_{\chi = 0} \quad (3)$$

$$\overset{(CE15.2)}{=} g_Q \sum_p \int \frac{dE}{e}\, T_p(E)\, [\tilde{f}_L(E) - f_R(E)] \quad (4)$$

TDT9

$$I_{dc} = \sum_{\ell} p_\ell \sum_{p} \int \frac{dE}{e} T_p(E) \left[f_F(E - eV - \hbar\Omega\ell) - f_R(E) \right] \quad (1)$$

$$= \sum_{\ell} p_\ell \, I(V + \hbar\Omega\ell/e) \quad (2)$$

current at DC-bias $V + \hbar\Omega\ell/e$

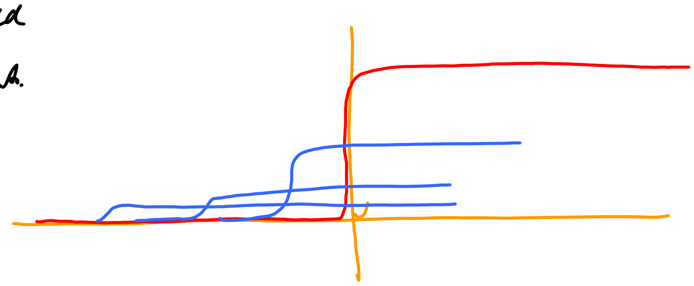
Similarly for zero-frequency noise:

noise at DC-bias $V - \hbar\Omega/e$

$$S_{dc} = \sum_{\ell} p_\ell \, S(V + \hbar\Omega\ell/e) \quad (3)$$

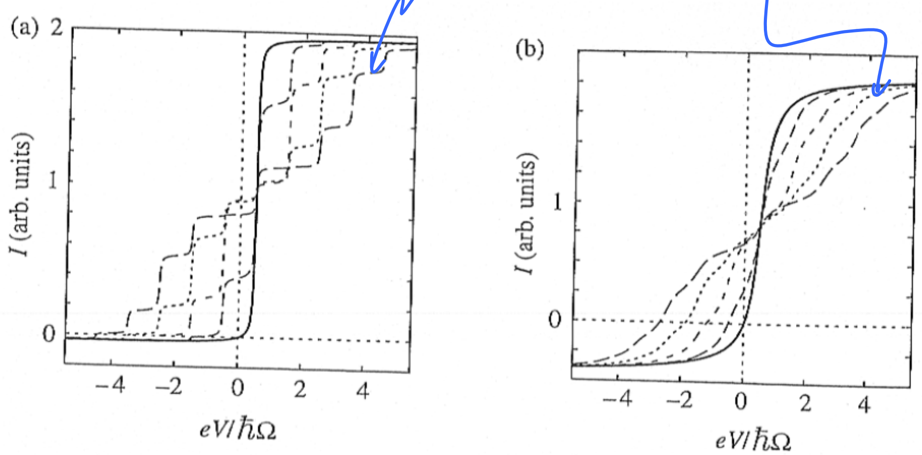
I_{dc}, S_{dc} are sums of shifted functions with decreasing weight. gives rise to step functions.

If $I(V)$ is linear $\sim V$, so is $I_{dc}(V)$ i.e. no steps.



TDT10

steps become smaller when $p_\ell \to 0$ for $\ell \to \infty$.



Tien–Gordon modification of dc I–V curves (thick solid lines). (a) Sharp step of dc I–V curve at $V_0 = 0.5\hbar\Omega$ is multiplied in the presence of ac voltage to be seen at any $V = V_0 + \hbar\Omega l/e$. Subsequent curves correspond to $e\tilde{V}/\hbar\Omega$, ranging from 0 to 4 with step 1. (b) The same curves fc a less sharp step. The peculiarities are barely seen and dc voltage mainly results in overall smoothing of the curve.

TDT11

$$a_\ell = J_\ell\left(\frac{e\tilde{V}}{\hbar\Omega}\right) = J_\ell(\tilde{V}/\tilde{V}_\Omega) \quad , \text{ where } \tilde{V}_\Omega = \hbar\Omega/e \qquad (1)$$

If $\tilde{V}/\tilde{V}_\Omega \ll 1$, $a_\ell \simeq 0$ for $\ell \neq 0$ few photons emitted or absorbed. (2)

If $\tilde{V}/\tilde{V}_\Omega \gg 1$, $\Rightarrow$ many photons, but (3)

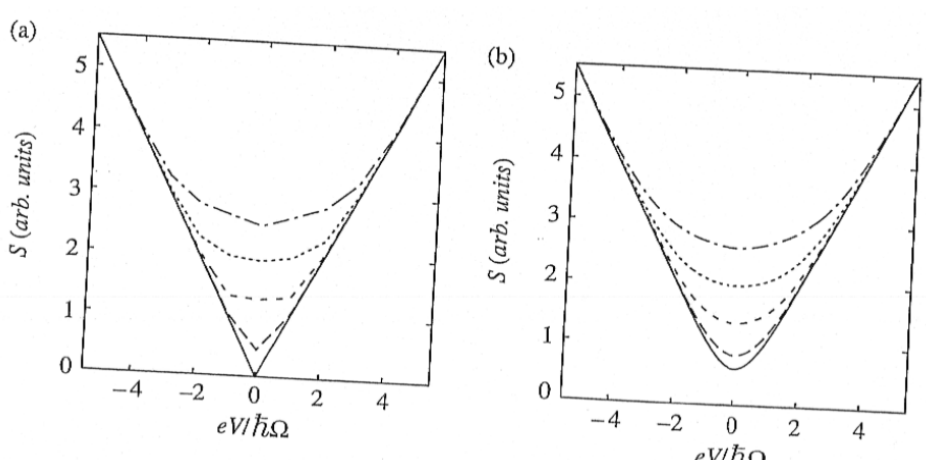
modulation slow relative to shift in $V(t) = V + \tilde{V}\sin\Omega t$

$\Rightarrow$ (9.2) reduces to its adiabatic limit, namely time-average over 1 period.

$$I_{dc} = \int_0^{2\pi/\Omega} dt \left(\frac{\Omega}{2\pi}\right) I(V + \tilde{V}\sin\Omega t) \qquad (4)$$

Rectification: I_{dc} can be $\neq 0$ even if $V = 0$, provided $I(V) \neq$ odd in V.
[for both (9.2) and (11.4)]. Dissipated power $I_{dc}V$ can be < 0 $\Rightarrow$
Power source gains energy (from ac EM-field) when transferring electron!

TDT12



Tien–Gordon modification of voltage-dependent noise. (a) For $k_B T \to 0$ the noise curve has a cusp at V=0. This cusp is multiplied in the presence of ac voltage. (b) If the cusp is smoothed by the temperature ($k_B T = \hbar\Omega$ is taken), the ac voltage just smoothes it further. This results in increased noise at zero dc voltage. The values of $e\tilde{V}/\hbar\Omega$ corresponding to different curves are the same as in Fig. 1.46.

1.7.4 Adiabatic Pumping (cyclic variation of $V_\alpha(t)$)

TDT 13

Charge transferred to transport channel j upon $\hat{S} \to \hat{S} + \delta\hat{S}$:

Ansatz:
(to be motivated below):

$$\delta\hat{Q}_j = \frac{-ie}{2\pi}\left(\delta\hat{S}\,\hat{S}^{\dagger}\right)_{jj} \tag{1}$$

or, if $S = S(E)$:

$$= \frac{-ie}{2\pi}\int dE \left[-\frac{\partial f(E)}{\partial E}\right]\left(\delta\hat{S}(E)\,\hat{S}^{\dagger}(E)\right)_{jj} \tag{2}$$

Consequence: Current induced in terminal α:

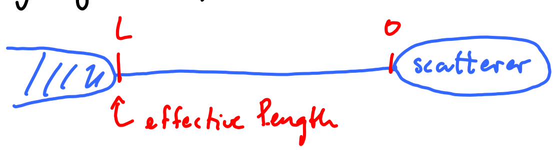
$$I_\alpha = \sum_j \frac{\delta\hat{Q}_j}{\delta t} \qquad \text{[sum over all channels } j \text{ in lead } \alpha\text{]} \tag{3}$$

$$= \frac{ie}{2\pi}\int dE\, \frac{\partial f(E)}{\partial E} \sum_\gamma \underbrace{\mathrm{Tr}}_{\text{over all channels}}\, \dot{\delta\hat{S}}_{\alpha\gamma}(E)\, \hat{S}^{\dagger}_{\gamma\alpha}(E) \tag{4}$$

Plausibility arguments for (13.1), (13.2):

TDT 14

I.



$$\hat{S} = \hat{r} = e^{i\theta} \tag{1}$$

Charge in channel: δQ = el. density $= e\,\delta L \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \underbrace{f(k_x)}_{|k_x|<k_F} = \frac{e k_F\,\delta L}{\pi}$ (2)

Wavefunction in incident channel:

$$\psi(x) = e^{-ik_F x} + e^{+ik_F x}\, e^{i\theta} \tag{3}$$

$\theta \to \theta + \delta\theta$ implies $L \to L + \delta L$:

$$\psi(L) = e^{-ik_F L}\left[1 + e^{+i2k_F L}\, e^{i\delta\theta}\, e^{i\theta}\right] \tag{4}$$

"size of scatterer changes" → "size of channel changes"

$$= \left[1 + e^{+i2k_F(L+\delta L)}\, e^{i\theta}\right] \tag{5}$$

$$\delta\theta \equiv 2k_F\,\delta L \tag{6}$$

$$\Rightarrow \quad \delta Q = \frac{e}{2\pi}\,\delta\theta = -\frac{ie}{2\pi}\,\delta S\, S^{\dagger} \quad \text{(cf. 13.1)} \tag{7}$$

II: Recover Landauer formula

TDT15

Voltages on lead α: $V_\alpha(t)$ (but small

$$\Rightarrow \quad \psi_\alpha(t) \xrightarrow{(4.5)} \psi_\alpha(t)\, e^{i\phi_\alpha}, \qquad \phi_\alpha = -\frac{e}{\hbar}\int^t dt'\, V_\alpha(t') \qquad (1)$$

$$\hat{S}_{\alpha\gamma} \longrightarrow e^{i(\phi_\alpha - \phi_\gamma)}\, \hat{S}_{\alpha\gamma} \qquad (2)$$

$$\dot{\hat{S}}_{\alpha\gamma} \longrightarrow \underbrace{-i\, e/\hbar\, (V_\alpha - V_\gamma)\, \hat{S}_{\alpha\gamma}} \qquad (3)$$

$$I_\alpha \overset{(13.4)}{=} \frac{ie}{2\pi}\int dE\, \frac{\partial f(E)}{\partial E} \sum_\gamma \mathrm{Tr}\left[\delta \dot{\hat{S}}_{\alpha\gamma}(E)\; \hat{S}^\dagger_{\gamma\alpha}(E)\right] \qquad (4)$$

$$= \underbrace{\int dE\left(-\frac{\partial f(E)}{\partial E}\right)}_{=1} \sum_\gamma \underbrace{(-)\, G_Q\, \mathrm{Tr}\left(\delta_{\alpha\gamma} - \hat{S}_{\alpha\gamma} S^\dagger_{\gamma\alpha}\right)}_{G_{\alpha\gamma}} V_\gamma \qquad (5)$$

$$= \sum_\gamma G_{\alpha\gamma} V_\gamma \quad = \text{multiterminal Landauer (MTC 4.4)} \qquad (6)$$

Pumping with two-terminal one-channel conductor:

TDT16

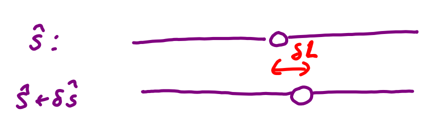
$$\hat{S} = \begin{pmatrix} \sqrt{R}\, e^{i\theta} & \sqrt{T}\, e^{i\eta} \\ \sqrt{T}\, e^{i\eta} & -\sqrt{R}\, e^{i(2\eta-\theta)} \end{pmatrix} \quad \text{for: } \begin{array}{l} R(t),\ T(t) = 1-R \\ \theta(t),\ \eta = \text{fixed.} \\ \delta T = -\delta R \end{array} \qquad (1)$$

$$\delta\hat{S} = \begin{bmatrix} \sqrt{R}\, e^{i\theta}\left(\frac{1}{2}\frac{\delta R}{R} + i\,\delta\theta\right) & \sqrt{T}\, e^{i\eta}\, \frac{1}{2}\frac{\delta T}{T} \\ \sqrt{T}\, e^{i\eta}\, \frac{1}{2}\frac{\delta T}{T} & -\sqrt{R}\, e^{i(2\eta-\theta)}\left[\frac{1}{2}\frac{\delta R}{R} - i\,\delta\theta\right] \end{bmatrix} \qquad (2)$$

$$\delta Q_L \overset{(13.1)}{=} \frac{-ie}{2\pi}\left(\delta\hat{S}\,\hat{S}^\dagger\right)_{LL} = \frac{e}{2\pi}\, R\,\delta\theta \qquad (3)$$

$$\delta Q_R = \frac{-ie}{2\pi}\left(\delta\hat{S}\,\hat{S}^\dagger\right)_{RR} = -\frac{e}{2\pi}\, R\,\delta\theta \qquad (4)$$

$\delta\theta \overset{(4.7)}{=} 2k_F\,\delta L$: scatterer shifts



$\delta Q_L = -\delta Q_R$

Algebra:

TDT17

$$\begin{bmatrix} \sqrt{R}\, e^{i\theta} \left(\frac{1}{2}\frac{\delta R}{R} + i\delta\theta\right) & \sqrt{T}\, e^{i\eta}\ \frac{1}{2}\frac{\delta T}{T} + i\delta\eta) \\ \sqrt{T}\, e^{i\eta}\left(\frac{1}{2}\frac{\delta T}{T} + i\delta\eta\right) & -\sqrt{R}\, e^{i(2\eta-\theta)}\left[\frac{1}{2}\frac{\delta R}{R} + i(2\delta\eta - \delta\theta)\right] \end{bmatrix} \cdot \begin{bmatrix} \sqrt{R}\, e^{-i\theta} & \sqrt{T}\, e^{-i\eta} \\ \sqrt{T}\, e^{-i\eta} & -\sqrt{R}\, e^{-i(2\eta-\theta)} \end{bmatrix}$$

LL: $\left(\frac{1}{2}\cancel{\delta R} + i\delta\theta R\right) + \left(\frac{1}{2}\cancel{\delta T} + i\delta\eta T\right) = i\left(\delta\theta R + \delta\eta T\right)$

RR: $\left(\frac{1}{2}\cancel{\delta T} + i\delta\eta T\right) + \left[\frac{1}{2}\cancel{\delta R} + i(2\delta\eta - \delta\theta)R\right] = -i\left[\delta\theta R + \delta\eta(T-2)\right]$

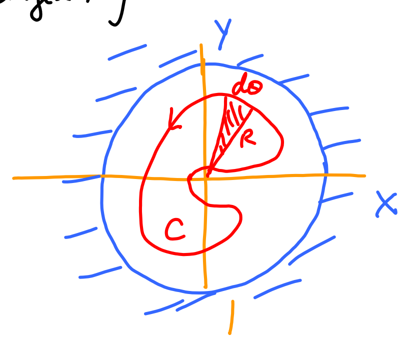
$i 2\delta\eta(1-T)$

TDT18

Let $R(t)$ and $\theta(t)$ vary in cyclic fashion, parametrized by

$$X = \sqrt{R}\cos\theta\,, \quad Y = \sqrt{R}\sin\theta\,, \qquad (1)$$

$$(\text{with } X^2 + Y^2 \le 1) \qquad (2)$$



Charge pumped per period:

$$Q_L \overset{(6.3)}{=} \int_0^{\text{Period}} dt\, \frac{dQ_L}{dt} = \frac{e}{2\pi}\int_0^{\text{Period}} dt\ R(t)\, \frac{d\,\delta\theta}{dt} \qquad (3)$$

$$= \frac{e}{2\pi}\oint_{\partial C \leftarrow \text{contour}} R\, d\theta = \frac{e}{\pi}\int_C dX\, dY = \frac{e}{\pi} A_C \qquad (4)$$

$\uparrow \pm$ area of contour swept counterclockwise/clockwise

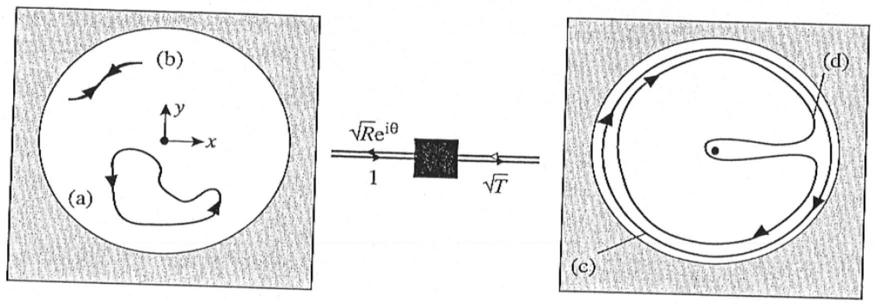
Net charge pumped has nice geometrical interpretation!

TDT19

Only contours are allowed for which pump action is cyclic, i.e.

$$\oint d\theta \overset{(14.6)}{=} \text{net displacement of scatterer} = 0$$

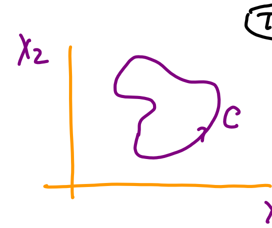
Moreover: enclosed area $\overset{(18.4)}{\neq 0}$ is required.



Different pumping cycles of a one-channel scatterer plotted in convenient coordinates (see the text). The net charge pumped over the period is given by the area enclosed by the closed contour (a) presenting the pumping cycle. The pumping requires variation of two independent parameters: contour (b), corresponding to a one-parameter cycle, encloses no area. Cycle (c) is forbidden because the corresponding contour encloses the origin (the cycle would give rise to a net shift of the scatterer). Cycle (d), which is very similar to cycle (c), is allowed.

TDT20

General case (multi-terminal)



Suppose $\quad S(t) = S(X_1(t), X_2(t)) \quad$ (1)

Transferred charge: for $\delta X_1, \delta X_2$: $\quad \delta \hat{Q}_j \overset{(13.1)}{=} \frac{-ie}{2\pi} (\delta\hat{S}\hat{S}^\dagger)_{jj} \quad$ (2)

$$= \frac{-ie}{2\pi}\left[\left(\frac{\partial \hat{S}}{\partial X_1}\hat{S}^\dagger\right)_{jj}\delta X_1(t) + \left(\frac{\partial \hat{S}}{\partial X_2}\hat{S}^\dagger\right)_{jj}\delta X_2(t)\right] \quad (3)$$

Transferred charge per cycle: $\quad Q_j^{\text{cycle}} = \oint \delta Q_j \quad$ (4)

Use Green's formula: $\quad \oint_{\partial C} \bar{F}\cdot d\vec{l} = \int dX_1 dX_2 \left(\frac{\partial F_2}{\partial X_1} - \frac{\partial F_1}{\partial X_2}\right) \quad$ (5)

TDT21

$$-\frac{i}{2}\left[\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right] = -\frac{i}{2}\left[\frac{\partial}{\partial x_1}\left(\frac{\partial \hat{S}}{\partial x_2}\hat{S}^\dagger\right)_{jj} - \frac{\partial}{\partial x_2}\left(\frac{\partial \hat{S}}{\partial x_1}\hat{S}^\dagger\right)_{jj}\right] \quad (1)$$

$$= -\frac{i}{2}\left[\left(\frac{\partial \hat{S}}{\partial x_2}\frac{\partial \hat{S}^\dagger}{\partial x_1}\right)_{jj} - \left(\frac{\partial \hat{S}}{\partial x_1}\frac{\partial \hat{S}^\dagger}{\partial x_2}\right)_{jj}\right] \quad (2)$$

$$= -\mathrm{Im}\left(\frac{\partial \hat{S}}{\partial x_1}\frac{\partial \hat{S}^\dagger}{\partial x_2}\right)_{jj} \equiv K_j(x_1, x_2) \quad (3)$$

$$Q_j^{cycle} \overset{(20.4)}{=} \frac{e}{\pi}\int_C dx_1 dx_2 \; K_j(x_1, x_2) \quad \text{(Brouwer formula)} \quad (4)$$

↖ area enclosed by pumping loop.

NICE GEOMETRIC INTERPETATION !!