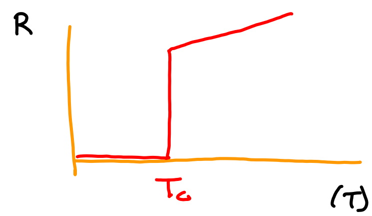
1.8. Andreev Scattering (AS) at normal-superconducting interfaces

AS1

Basic facts about SC (Appendix B)
(according to BCS theory)



- resistance vanishes for $T < T_c$
- Meissner effect: magnetic fields do not penetrate bulk s.c.
- Reason: electrons form Cooper pairs (1)
- Attraction arises via electron-phonon interaction: $V_{e\text{-}ph} = -\frac{2\lambda}{\nu}\,\delta(\vec{r}_1 - \vec{r}_2)$

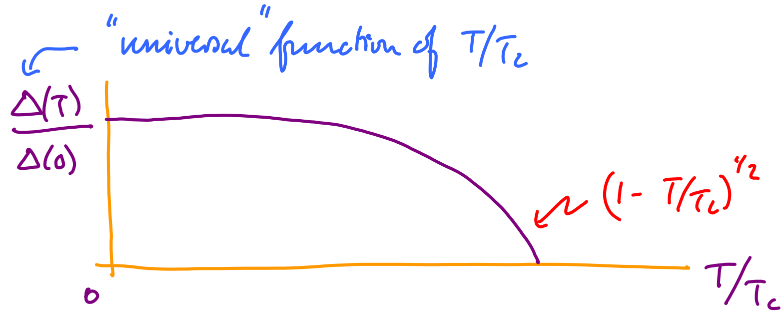
  ← electron DOS

- "Binding energy" of Cooper pair: $\Delta = \begin{cases} \hbar\omega_D\, e^{-1/\lambda} = 1.76\, k_B T_c & \text{(at } T=0) \quad (2) \\ (T_c - T)^{1/2} & (\text{as } T \to T_c) \quad (3) \end{cases}$

  ω_D ← typical phonon freq.

  = "sc energy gap"

AS2

- 

- Order parameter $= \Delta e^{i\varphi}$ = complex number! (1)
- supercurrent: $\vec{j}_s = e n_s (\vec{v}_s - e/mc\, \vec{A})$ (2)

  superfluid density: n_s $(\sim \Delta)$ (3)

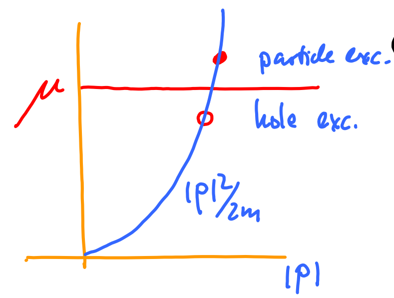
  superfluid velocity: $\vec{v}_s = \frac{\hbar}{2m} \vec{\nabla}\varphi$ (4)

- In bulk SC: $\vec{\nabla}\varphi = 0$, no supercurrent
- At boundary between two SC's: if $\varphi_1 - \varphi_2 \neq 0 \Rightarrow \vec{j}_s \neq 0 \Rightarrow$ Josephson effect (5)

AS3

- Excitation spectrum of normal metal:

$$\xi_p \overset{(e.g.)}{=} \frac{p^2}{2m} - \varepsilon_F \qquad (1)$$

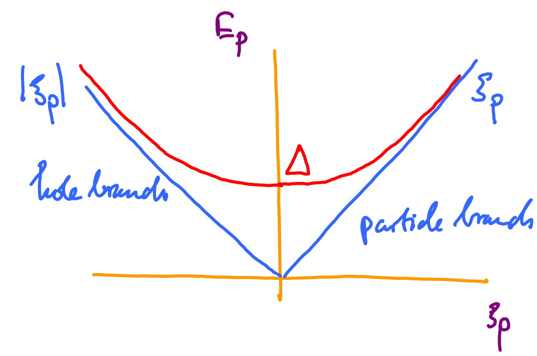


Extra particle above Fermi sea has energy $\xi_e(\vec{p}) \equiv \xi_{\vec{p}} > 0 \qquad (2)$

hole = missing particle below " " " " $\xi_h(\vec{p}) = -\xi_{\vec{p}} > 0 \qquad (3)$

Excitation spectrum of SC:

$$E_p = \sqrt{\Delta^2 + \xi_p^2}$$



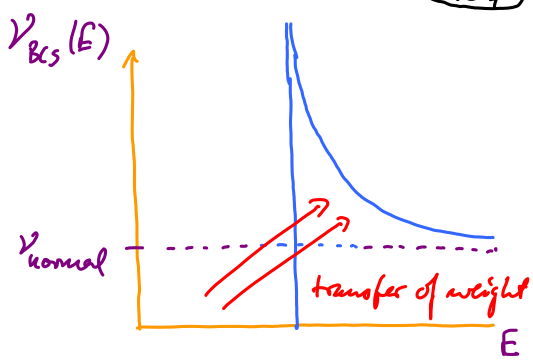
No excitations in SC. with $E < \Delta$

AS4

- BCS density of states:

$$\nu_{BCS}(E) = \sum_{\vec{p}} \delta(E - E_p) \qquad (1)$$

$$\propto \frac{\theta(E-\Delta)}{\sqrt{E-\Delta}} \qquad (2)$$



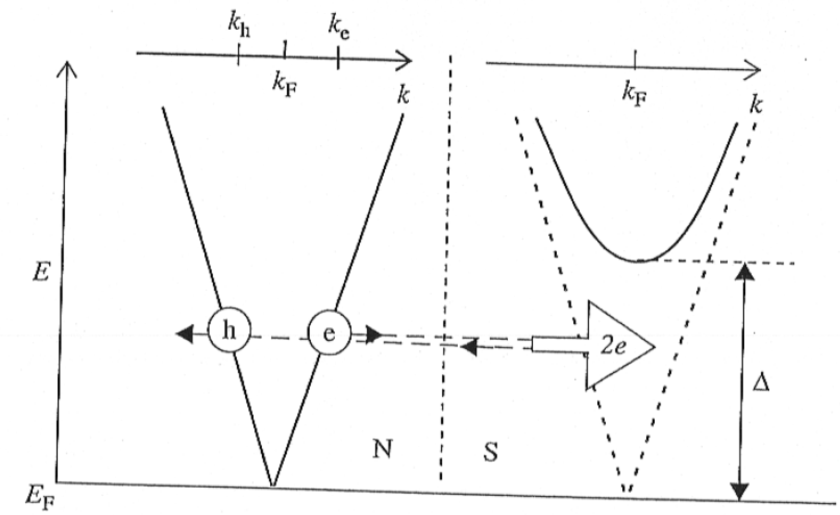
- "Size of Cooper pair" = "Superconducting correlation length"

$$\xi \equiv \frac{\hbar v_F}{\Delta} = 1\,\mu m \qquad \left[\begin{array}{ll} \text{for } v_F \simeq 10^6\ m/s \\ \frac{\Delta}{k_B} \simeq 10\ K \end{array} \right] \qquad (3)$$

$$= \text{huge!}$$

1.8.1 Andreev reflection (1964)

AS5

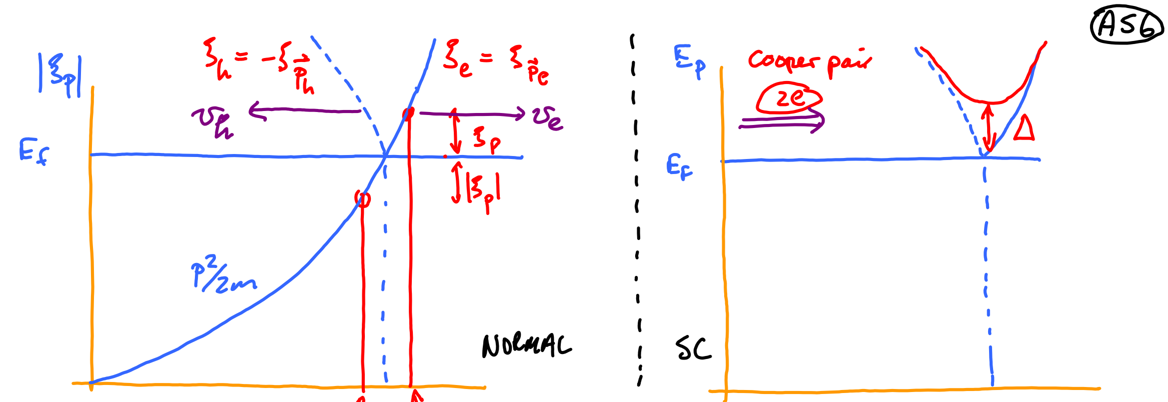


Andreev reflection: an electron coming from a normal (N) metal to a superconductor (S) is reflected as a hole with the same energy and approximately the same momentum.

Energy is conserved

charge 2e is transferred from N → S

AS6



since $|\xi_p|/E_F \ll 1$: $\Leftarrow$ $\begin{cases} p_e = p_F + \xi_p/v_F & (1) \\ p_h = p_F - \xi_p/v_F & (2) \end{cases}$

$p_e \simeq p_F \simeq p_h$ (3)

Velocity of electron excitation: $\vec{v}_e = \dfrac{\partial \xi_e(\vec{p})}{\partial \vec{p}_e} \simeq \vec{p}_e/m$ (4)

hole excitation: $\vec{v}_h = \dfrac{\partial \xi_h(\vec{p})}{\partial \vec{p}_h} \simeq -\dfrac{\vec{p}_h}{m}$ (5)

opposite!

Andreev reflection:

AS7

Bogoliubov-de Gennes (BdG) equation for $\left.\begin{matrix}\psi_e(\vec r)\\ \psi_h(\vec r)\end{matrix}\right\}$ $\begin{matrix}\text{particle}\\ \text{hole}\end{matrix}$ wave-fct. (1)

(Schrödinger eq. for single-particle exc. in SC)

$$\begin{pmatrix} \hat H & \Delta e^{i\varphi} \\ \Delta e^{-i\varphi} & -\hat H \end{pmatrix} \begin{pmatrix} \psi_e(\vec r) \\ \psi_h(\vec r) \end{pmatrix} = E \begin{pmatrix} \psi_e(\vec r) \\ \psi_h(\vec r) \end{pmatrix} \qquad (2)$$

SC mixes electrons & holes

$$\hat H = \hat H_0 - E_F \, ,$$

$$\hat H_0 = -\frac{\hbar^2}{2m}\left(\vec\nabla + \frac{ie}{\hbar c}\vec A\right)^2 + U(\vec r)$$ (Hamilton of Normal metal)

In general: Δ and φ depend on $\vec r$; here, take them constant.

BdG at $\Delta = 0$, $U(\vec r) = 0$, $\vec A = 0$

AS8

Solutions: $$\begin{pmatrix} \psi_e \\ \psi_h \end{pmatrix} \sim \begin{pmatrix} \psi_e(0)\, e^{i\vec p_e\cdot\vec r/\hbar} \\ \psi_h(0)\, e^{i\vec p_h\cdot\vec r} \end{pmatrix} \qquad (1)$$

inserted into (7.2):

$$E = \frac{p_e^2 - p_F^2}{2m} = (p_e - p_F)\underbrace{\frac{(p_e + p_F)}{2m}}_{\simeq v_F} \Rightarrow p_e = E/v_F + p_F \qquad (2a)$$

$$E = -\frac{p_h^2 + p_F^2}{2m} = (p_F - p_h)\underbrace{\frac{(p_F + p_h)}{2m}}_{\simeq v_F} \Rightarrow p_h = -E/v_F + p_F \qquad (2b)$$

(2) implies:

For $E > 0$: $p_e > p_F$, $p_h < p_F$ (agrees with usual convention) (3)

For $E < 0$: $p_e < p_F$, $p_h > p_F$ (disagrees " " " ") (4)

(4) and (3) are related by $\psi_e \leftrightarrow \psi_h$:

so, retain only $E > 0$ solutions, for which $E_p = |\xi_p|$

BdG at $\Delta \neq 0$ $(\mathcal{U}=0, \vec{A}=0)$

AS9

Ansatz for solution:

$$\begin{pmatrix} \psi_e(\vec{r}) \\ \psi_h(\vec{r}) \end{pmatrix} = e^{i\vec{p}\cdot\vec{r}/\hbar} \begin{pmatrix} \psi_e(0) \\ \psi_h(0) \end{pmatrix} \quad (1)$$

(7.2) gives:

$$\xi_p = \frac{p^2 - p_F^2}{2m} = (p - p_F) v_F$$

$$\begin{pmatrix} \xi_p & \Delta e^{i\varphi} \\ \Delta e^{-i\varphi} & -\xi_p \end{pmatrix} \begin{pmatrix} \psi_e(0) \\ \psi_h(0) \end{pmatrix} = E \begin{pmatrix} \psi_e(0) \\ \psi_h(0) \end{pmatrix} \quad (2)$$

Diagonalize:

$$(\xi - E)(-\xi - E) - \Delta^2 = 0$$
$$E^2 - \xi^2 - \Delta^2 = 0 \quad (3)$$

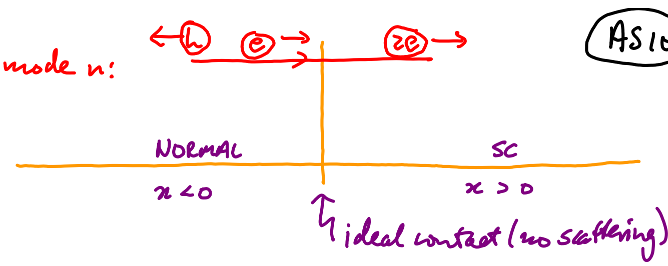
$\Rightarrow$ $$E = \sqrt{\Delta^2 + \xi_p^2} \quad \text{(retain only } E > 0 \text{ solution)} \quad (4)$$

$\Rightarrow$ In bulk, quasiparticles with $E < \Delta$ do not exist.

Ideal N-S contact

AS10

mode n:



Ansatz for solution:

$$\begin{pmatrix} \psi_e(x) \\ \psi_h(x) \end{pmatrix} = e^{i p_F^{(n)} x/\hbar} \begin{pmatrix} \tilde{\psi}_e(x) \\ \tilde{\psi}_h(x) \end{pmatrix} \quad (1)$$

describes electron moving to right and hole moving to left (see 8.2)

(7.1):

$$\hat{H}\, e^{i p_F^{(n)} x/\hbar}\, \tilde{\psi} = e^{i p_F^{(n)} x} \left[\frac{1}{2m} \left(\cancel{p_F^{(n)}} - i\hbar \partial_x \right) \left(\cancel{p_F^{(n)}} - i\hbar\partial_x \right) - \cancel{\frac{p_F^2}{2m}} \right] \tilde{\psi} \quad (2)$$

$$\simeq e^{i \tilde{p}_F^{(n)} x} \left[\left(-i\hbar v_F \partial_x \right) - \frac{\hbar^2 \partial_x^2}{2m} \right] \tilde{\psi} \quad (3)$$

Neglect $\partial_x^2 \tilde{\psi}$, assuming $\tilde{\psi}$ varies slowly compared to λ_F.

(10.3) in (7.2): **AS11**

$$\begin{pmatrix} -i\hbar v_F \partial_x & \Delta(x)\, e^{i\varphi} \\ \Delta(x)\, e^{-i\varphi} & i\hbar v_F \partial_x \end{pmatrix} \begin{pmatrix} \tilde\psi_e(x) \\ \tilde\psi_h(x) \end{pmatrix} = E \begin{pmatrix} \tilde\psi_e(x) \\ \tilde\psi_h(x) \end{pmatrix} \qquad (1)$$

For normal metal ($x<0$, $\Delta=0$):

$$\tilde\psi(x<0) = \begin{pmatrix} e^{ixE/v_F\hbar} \\ r_A\, e^{-ixE/v_F\hbar} \end{pmatrix} \quad \begin{matrix}\text{incident electron} \\ \text{reflected hole}\end{matrix} \qquad (2)$$

r_A: Andreev reflection amplitude

For SC ($x>0$, $\Delta \neq 0$):

$E<\Delta$ has only evanescent solution:

$$\tilde\psi(x>0) = C \begin{pmatrix} f_e \\ f_h \end{pmatrix} e^{-x\kappa}, \quad \text{with } \kappa = \frac{\sqrt{\Delta^2-E^2}}{\hbar v_F} \qquad (3)$$

$|f_e|^2 + |f_h|^2 = 1$

Check: $(+i\hbar v_F\kappa - E)(-i\hbar v_F\kappa - E) - \Delta^2 = 0.$

$$\Rightarrow (\hbar v_F)^2\kappa^2 + E^2 - \Delta^2 = 0 \quad \Rightarrow \quad \kappa = \pm\sqrt{\Delta^2 - E^2}/\hbar v_F \qquad (4)$$

AS12

Scale of penetration into SC:

$$\xi = \frac{1}{\kappa} = \frac{\hbar v_F}{\sqrt{\Delta^2 - E^2}} \qquad (1)$$

$$\begin{cases} \gg \lambda_F = \frac{\hbar v_F}{E_F} & \text{for } E = 0 \\ \to \infty & \text{for } E \to \Delta \end{cases} \qquad (2)$$

Match solutions (11.2) & (11.3) at $x=0$: $\tilde\psi(x\to 0^-) = \tilde\psi(x\to 0^+)$

($\partial_x\psi$ does not have to be continuous, since (11.1) contains only 1st. derivatives)

One finds:
$$r_A = e^{-i\varphi}\left(\frac{E}{\Delta} - i\frac{\sqrt{\Delta^2-E^2}}{\Delta}\right) \equiv e^{i\chi} \qquad (3)$$

$$\chi = -\arccos(E/\Delta) - \varphi \qquad (4)$$

$\Rightarrow \quad |r_A|^2 = 1 \quad \Rightarrow$ electron is fully reflected as a hole, with relative phase shift χ

Similarly: incident hole is reflected as electron,

with $|r_A| = 1$ $\qquad \tilde\chi = -\arccos(E/\Delta) + \varphi$

AS13

For $E > \Delta$:

$$\tilde{\psi}(x>0) = c \begin{pmatrix} f_e \\ f_h \end{pmatrix} e^{ikx} \quad , \text{ with } k = \sqrt{E^2 + \Delta^2}/v_F\hbar \qquad (1)$$

$$r_A = e^{-i\varphi}\left(\frac{E}{\Delta} - \frac{\sqrt{E^2 - \Delta^2}}{\Delta}\right) \qquad (2)$$

$$|r_A|^2 < 1 \quad \Rightarrow \quad \text{single electrons can penetrate SC} \qquad (3)$$

$$r_A \rightarrow e^{-i\varphi}\,\frac{1}{2}\,\frac{\Delta}{E} \quad \text{for } E/\Delta \gg 1 \qquad (4)$$