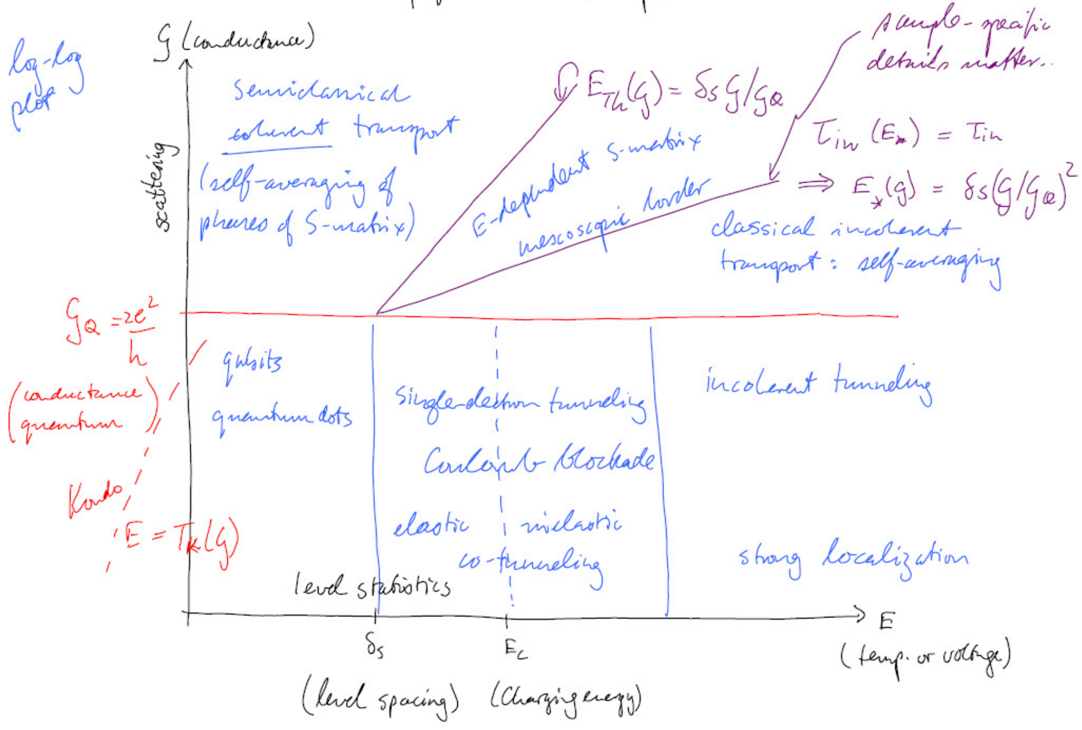
"Map of Quantum Transport"



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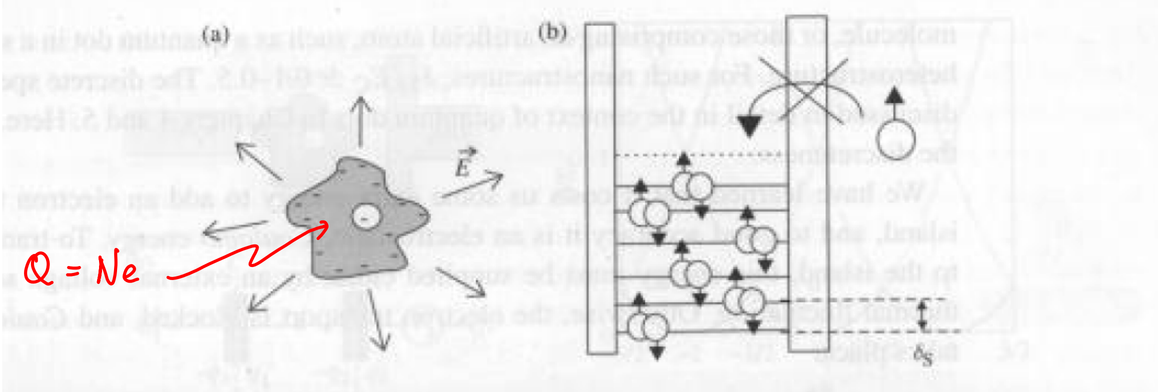
3. Coulomb Blockade (CB)

3.1 Charge Quantization and Charging Energy (CE)



(a) Excess charge in an isolated metallic island produces an electric field outside, thus accumulating charging energy. (b) The energy cost to put an electron into the island is not just a typical level spacing δ_S: it includes charging energy $E_C \gg \delta_S$.

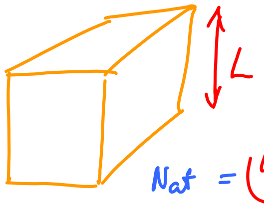
Electrostatic energy of charged island with N electrons :

$$E(N) = \frac{Q^2}{2C} = \underbrace{\frac{e^2}{2C}}_{\equiv E_c \,=\, \text{"charging energy"}} N^2 = E_c N^2 \qquad (1)$$

"Addition energy" that must be provided to add (N+1)st electron:

Coulomb: $E_{N+1} - E_N = E_c\left((N+1)^2 - N^2\right) = E_c(2N+1)$ (1)

Estimate scales of cubic island of size, with one valence electron per atom.



$N_{at} = (L/a)^3$

lattice spacing

Finite mean level spacing: $\delta_s = \frac{E_F}{N_{at}}$

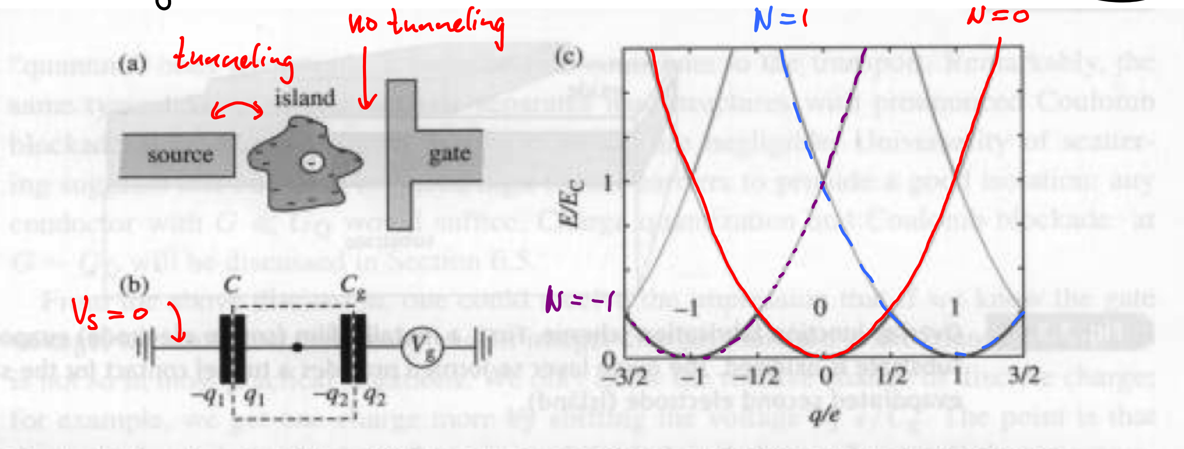
Coulomb's law estimate: (charge spread over distance L) $E_c \simeq e^2/L$

$$\frac{\delta_s}{E_c} = \left(\frac{E_F}{N_{at}}\right)\left(\frac{L}{e^2}\right) = \left(\underbrace{\frac{E_F a}{e^2}}_{\simeq 1}\right)\left(\frac{L}{aN_{at}}\right) \simeq \frac{1}{(N_{at})^{2/3}} \rightarrow 0$$

in limit of many atoms

(Since typically $\varepsilon_F \simeq e^2/a$)

3.1.1 Single electron box



Single-electron box. (a) Setup. (b) Equivalent capacitance circuit (the dashed box includes capacitance plates that belong to the island). (c) The charging energy of the single-electron box versus $q = -C_gV_g$. Each parabola corresponds to a charge state with *N* excess charges. The lowest segments of the parabolas give the minimum energy and the actual charge state.

Electrostatic energy: $E_{el} = \frac{1}{2}\frac{q_1^2}{C} + \frac{q_2^2}{C_g} - V_g q_2$ (1)

↑ energy to charge capacitors

↑ energy to transfer charge q_2 to gate electrode

CB5

Use $q_1 = CV_1$, with voltage drops V_1 , V_2

$q_2 = C_g V_2$ across capacitors C, C_g. (1)

$$E_{el} = \frac{1}{2}\left(CV_1^2 + C_g V_2^2\right) + \underbrace{\left(-V_g C_g\right)}_{\equiv q} V_2$$ "charge induced on island by gate" (not discrete) (2)

Voltage drop across island: $V_g = V_1 + V_2$ (3)

Charge quantization on island: $eN = q_1 - q_2 = CV_1 + C_g V_2$ (4)

Eliminate V_2: $C_g V_1 \underbrace{- C_g V_g}_{q} \overset{C_g \cdot (3)}{=} C_g V_2 \overset{(4)}{=} eN - CV_1 \Rightarrow V_1 = \frac{eN - q}{C + C_g}$ (5)

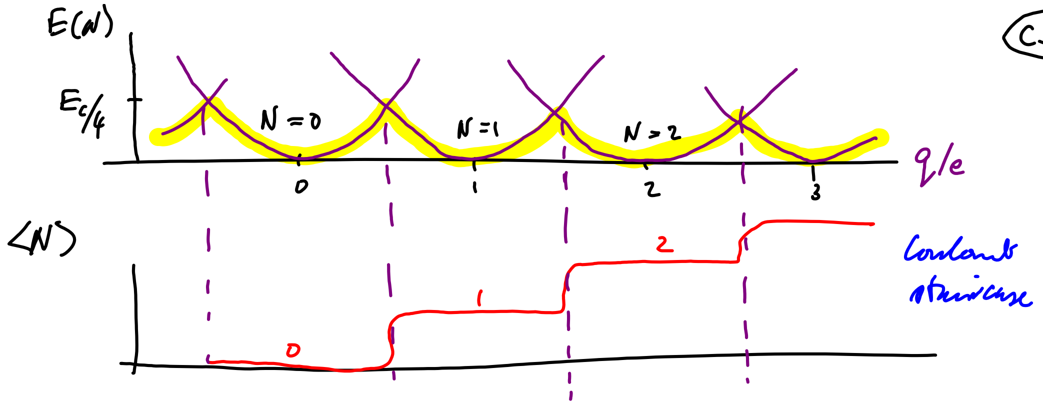
(3) into (2)

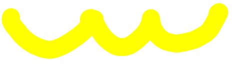
$$E_{el} = \frac{1}{2}\left(C + C_g\right) V_1^2 + \frac{1}{2} C_g V_g^2 \qquad (6)$$

$$\boxed{E_{el} \overset{(5)}{=} E_c\left(N - q/e\right)^2 - \frac{q^2}{2C_g}, \quad (7) \qquad E_c = \frac{e^2}{2(C + C_g)}} \qquad (8)$$

e-Periodic in q, due to discreteness of charge !! $\equiv$ "charging energy"

CB6

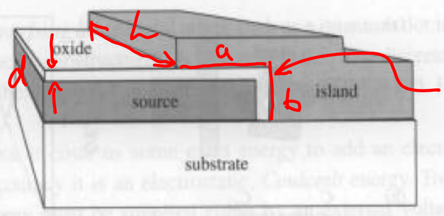


- E(N) minima at : $q_N = Ne$
- Ground state has N electrons for $q \in [q_N - \frac{1}{2}e, q_N + \frac{1}{2}e]$
- Minimum energy  is periodic in q, with period e
- If $E_c = 0$ (e.g. due to a short), then minimum energy would be

  E(N) =  = 0 = flat.

3.1.2 Islands & Barriers

tunnel junction for
metal-based
CB system:



Overlap junction fabrication scheme. First, a metallic film (source electrode) evaporated on the substrate is oxidized. The oxide layer so formed provides a tunnel contact for the subsequently evaporated second electrode (island).

Geometric capacitance : $\sim L$

Area: $A = L(a+b)$

Capacitance : $C \sim A/d \gg L$

- (too) weak contacts : $\Rightarrow$ good isolation, large E_c, good charge quantization
  (thick barriers; small area) $\Rightarrow$ (too) small current (to measure)
- (too) good contacts : $\Rightarrow$ (too) poor isolation, small E_c, weak charge quant.
  (thin barriers; large area) $\Rightarrow$ large current

What is optimal choice for barrier strength ?

Requirements on tunnel resistance $(R_T \equiv \frac{1}{G_T})$ too see CB:

Add extra electron at energy cost : E_c (1)

It will escape within RC-time : $\tau_{RC} = R_T C = \frac{C}{G_T}$ (2)
(classical discharge time for capacitor)

Heisenberg uncertainty: $\Delta E \Delta t \geq \hbar$ (3)

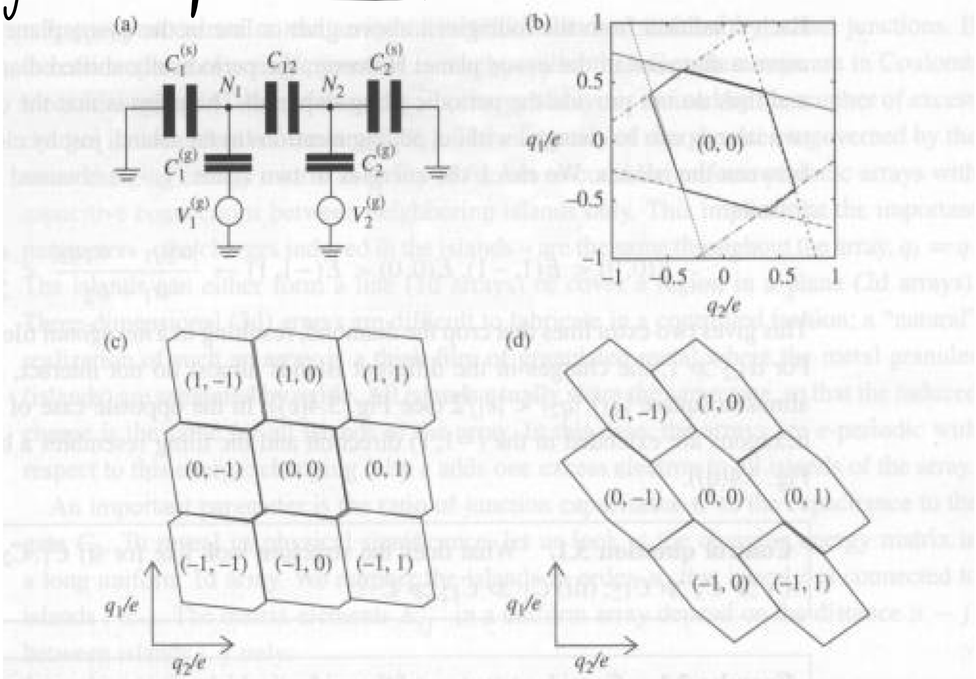
implying energy uncertainty of: $\Delta E \simeq \frac{\hbar}{\tau_{RC}} \simeq \frac{\hbar G_T}{C}$ (4)

For state with extra electron to be well-defined, we need : $\frac{\hbar G_T/\cancel{C}}{e^2/\cancel{C}} = \frac{\Delta E}{E_c} \ll 1$ (5)

$$\boxed{G_T \ll \frac{e^2}{h} = G_Q} \quad (6)$$

High tunnel barriers not needed to see CB! $G_T \ll G_Q$ is sufficient.

3.1.3 Many-island capacitive circuits



(a) Equivalent capacitance circuit of a two-island system. (b) The energies of two different charge states are equal along a line in the q_1–q_2 plane. Six lines define a region where the state (0, 0) has minimum energy. (c), (d) Periodic tilings of the q_1–q_2 plane.

Preview: Interactions between electrons on different islands (i) leads to:

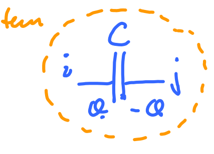
$$E_{el} = \sum_{ij} \underbrace{E^{(c)}_{ij}}_{} (N_i - q_i/e)(N_j - q_j/e) \qquad (1)$$

→ charging energy matrix ~ $\dfrac{1}{\text{capacitance matrix}}$

- E_{ii} : diagonal elements : cost to add charge to island i
- $E_{i\neq j}$: interaction between charges on different islands

Elementary interlude:

Charging a capacitor



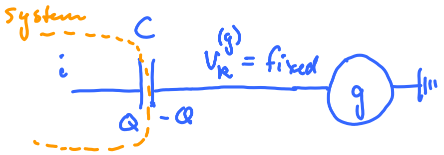
$$E^{(c)}_{ij}(Q) = \int_0^Q dQ' \underbrace{[V_i - V_j](Q')}_{CQ'} \qquad (1)$$

$V_i - V_j = CQ$

$$= \frac{1}{2} C Q^2 = \frac{1}{2} \frac{V(Q)^2}{C} \qquad (2)$$

charging a capacitor attached to a gate at fixed voltage

$$\overset{(k)}{\underset{ik}{E}}(Q) = \int_0^Q dQ' \underbrace{V_i(Q')}_{CQ'+V_k^{(g)}} \qquad (1)$$



$$= \frac{1}{2} C Q^2 + Q V_k^{(g)} \qquad (2)$$

$$V_i - V_k^{(g)} = CQ$$

$$= \frac{1}{2} \frac{(V_i - V_k^{(g)})^2}{C} - \left(-Q V_k^{(g)}\right) \qquad (3)$$

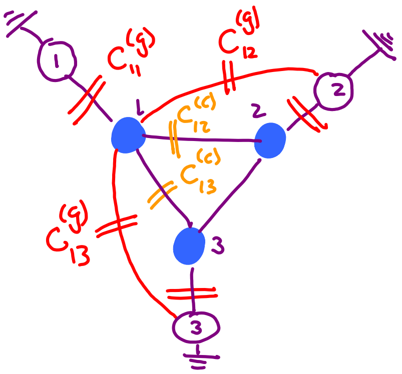
Interpretation: system looses energy $(-QV_k^{(g)})$ in order to put $-Q$ on gate

Many islands:

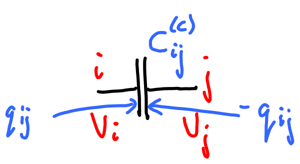
label i, charge eN_i, voltage on island, V_i

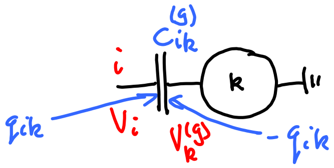
Gate electrodes (including source electrodes):
label k, voltage $V_k^{(g)}$,

Capacitance between
- islands i and j: $C_{ij}^{(c)}$ $(= 0 \text{ for } i = j)$
- island i and gate k: $C_{ij}^{(g)}$



By definition, $C_{ij}^{(c)}$ or $C_{ik}^{(g)}$ gives the induced charge at island i due to a voltage difference between i and j or i and k:





$$q_{ij} = C_{ij}^{(c)}(V_i - V_j) \qquad (1a)$$

$$q_{ik} = C_{ik}^{(g)}(V_i - V_k^{(g)}) \qquad (1b)$$

energy in capacitors (10.2) (11.3)

energy lost by system in order to put charge $-q_{ik}$ on gate ↓ (11.3)

$$E_{el} = \frac{1}{2}\sum_{i>j} \underset{①}{C_{ij}^{(c)}(V_i - V_j)^2} + \underbrace{\frac{1}{2}\sum_{i,k} \underset{②}{C_{ik}^{(g)}(V_i - V_k^{(g)})^2} - \sum_{i,k} \underset{③}{(-q_{ik})V_k^{(g)}}}_{\overset{\|}{(1b)}} \qquad (2)$$

$$\frac{1}{2}\sum_{ik} C_{ik}^{(g)} V_i^2 - \underbrace{\frac{1}{2}\sum_{ik} C_{ik}^{(g)} \left(V_k^{(g)}\right)^2}_{\text{ignore, since } N_i\text{-independent}} \qquad (3)$$

Total charge on island i: $$eN_i = \sum_j q_{ij} + \sum_k q_{ik}$$ (1) CB13

$$= \sum_j C^{(c)}_{ij}(V_i - V_j) + \sum_k C^{(g)}_{ik}(V_i - V^{(g)}_k)$$ (2)

$$N_i - \underbrace{\left(-\sum_k C^{(g)}_{ik} V^{(g)}_k\right)}_{\equiv q_i \ (3')} = \underbrace{\left[\sum_j C^{(c)}_{ij} + \sum_k C^{(g)}_{ik}\right]}_{\equiv C_{ii}} V_i + \sum\underbrace{\left(-C^{(c)}_{ij}\right)}_{\equiv C_{ij} \text{ for } i \neq j} V_j$$ (3)

offset charge induced on island i due to all gates

$$\boxed{eN_i - q_i \equiv \sum_j C_{ij} V_j}$$ (4a) with inverse $V_i = \sum_j (C^{-1})_{ij}(eN_j - q_j)$ (4b)

Capacitance matrix: $$C_{ij} = \begin{cases} \sum_i C^{(c)}_{ii} + \sum_k C^{(g)}_{ik} & \text{for } i = j \\ -C^{(c)}_{ij} & \text{for } i \neq j \end{cases}$$ (5) (recall: $C^{(c)}_{ii} = 0$)

CB14

$$E^{(1.2)}_{el} = \frac{1}{2}\sum_{i>j} C^{(c)}_{ij}(V_i - V_j)^2 + \frac{1}{2}\sum_{ik} C^{(g)}_{ik} V_i^2 - \cancel{\frac{1}{2}\sum_{ik} C^{(g)}_{ik}(V^{(g)}_k)^2}$$ ignore (1)

$$= \frac{1}{2}\left[\sum_{i>j} C^{(c)}_{ij} V_i^2 + \underbrace{\sum_{i>j} C^{(c)}_{ij} V_j^2}_{\sum_{j>i} C^{(c)}_{ji} V_i^2} - \underset{+\sum_{j>i} C^{(c)}_{ji} V_j V_i}{\cancel{2}\sum_{i>j} C^{(c)}_{ij} V_i V_j}\right] + \frac{1}{2}\sum_{ik} C^{(g)}_{ik} V_i^2$$ (2)

relabel in some terms: $i \leftrightarrow j$:

[recall: $C^{(s)}_{ij} = C^{(s)}_{ji}$]

Use: $$\left(\sum_{j>i} + \sum_{j<i}\right)\left(C^{(c)}_{ij}\right) = \sum_j C^{(c)}_{ij}$$ with no restriction on j-sum (since $C^{(c)}_{ii} = 0$) (3)

$$E_{el} = \frac{1}{2}\left[\sum_i V_i \underbrace{\left(\sum_i C^{(c)}_{ii} + \sum_k C^{(g)}_{ik}\right)}_{(12.5)\ = C_{ii}} V_i + \sum_{ji} V_i \underbrace{\left(-C^{(c)}_{ij}\right)}_{(12.5)\ = C_{ij}} V_j\right] = \frac{1}{2}\sum_{ij} \underbrace{V_i}_{(eN-q)C^{-1}} C_{ij} \underbrace{V_j}_{(12.4b)\ C^{-1}(eN-q)}$$ (4)

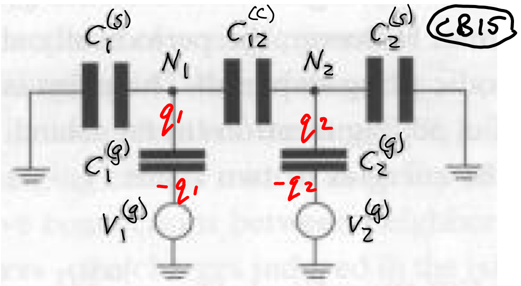
$$(eN-q)\underbrace{C^{-1} \cdot C \cdot C^{-1}}_{C^{-1}}(eN-q)$$

$$\boxed{E_{el} = \frac{1}{2}\sum_{ij} \underbrace{e^2 (C^{-1})_{ij}}_{E^{(c)}_{ij}} (N_i - q_i/e)(N_j - q_j/e)}$$ (5)

$E^{(c)}_{ij}$ = charging energy matrix

CB15

Example: two islands



$i, j, k = 1, 2$

Assume: $C^{(g)}_{i\neq k} = 0$ (1)

Write: $C^{(g)}_{ii} = C_i^{(g)}$ or $C_i^{(s)}$ (2)

Induced charges: $q_i \overset{(13.3')}{=} -\sum_k C^{(g)}_{ik} V_k^{(g)} = -C_i^{(g)} V_i^{(g)}$ (3)

capacitance matrix: $C_{ii} = \sum_{\bar{i}} C^{(c)}_{i\bar{i}} + \sum_k C^{(g)}_{ik} = C_{12}^{(c)} + C_i^{(g)} + C_i^{(s)} \equiv C_i \equiv C_{12}(1+\alpha_i)$ (4)

$C_{ij} = -C_{ij}^{(c)}$ (5) with $\alpha_i > 0$

$$\hat{C} = \begin{bmatrix} C_1 & -C_{12}^{(c)} \\ -C_{12}^{(c)} & C_2 \end{bmatrix} \Rightarrow \hat{E}_c = \frac{e^2}{2}\hat{C}^{-1} = \frac{e^2/2}{\underbrace{\left[C_1C_2 - (C_{12}^{(c)})^2\right]}_{\equiv E_2/C_{12}^{(c)}\ (7)}}\begin{bmatrix} C_2 & C_{12}^{(c)} \\ C_{12}^{(c)} & C_1 \end{bmatrix} \overset{(4)}{\equiv} E_2\begin{pmatrix} 1+\alpha_2 & 1 \\ 1 & 1+\alpha_1 \end{pmatrix} \quad (6)$$

CB16

Then

$$E_{el}(N_1, N_2) = \sum_{ij} (\hat{E}_{ij})(N_i - q_i/e)(N_j - q_j/e) \quad (1)$$

$$= E_2\left[(1+\alpha_2)(N_1 - q_1/e)^2 + (1+\alpha_1)(N_2 - q_2/e)^2 + 2(N_1 - q_1/e)(N_2 - q_2/e)\right] \quad (2)$$

$$= E_2\left[\alpha_2(N_1 - q_1/e)^2 + \alpha_1(N_2 - q_2/e)^2 + (N_1 + N_2 - q_1/e - q_2/e)^2\right] \quad (3)$$

e-Periodicity: if at given (q_1, q_2) ground state has charges (N_1, N_2),
then at $(q_1 + eM_1, q_2 + eM_2)$ " " $(N_1 + M_1, N_2 + M_2)$ (4)
and looks like original state w.r.t. its charging properties.

⇒ regions of definite ground state charge form periodic tiling of q_1, q_2 plane.
(see p. 9)

⇒ need to find only shape of tile at $(N_1, N_2) = 0$. (5)

CB17

$$E(N_1,N_2) = E_2\left[\alpha_2(N_1-q_1/e)^2 + \alpha_1(N_2-q_2/e)^2 + (N_1+N_2-q_1/e-q_2/e)^2\right] \quad (1)$$

$$E(0,0) = \frac{E_2}{e^2}\left[\alpha_2 q_1^2 + \alpha_1 q_2^2 + (q_1+q_2)^2\right] \quad (2)$$

$$E(\pm 1,0) = E(0,0) + E_2\left[\alpha_2 \mp 2\alpha_2 q_1/e + 1 \mp 2(q_1+q_2)/e\right] \quad (3)$$

$$E(0,\pm 1) = E(0,0) + E_2\left[\alpha_1 \mp 2\alpha_1 q_2/e + 1 \mp 2(q_1+q_2)/e\right] \quad (4)$$

$$E(\pm 1,\mp 1) = E(0,0) + E_2\left[\alpha_2 \mp 2\alpha_2 q_1/e + \alpha_1 \pm 2\alpha_1 q_2/e\right] \quad (5)$$

$$E(0,0) < E(\pm 1,0) \Rightarrow [\;]_{(3)} > 0 \Rightarrow \pm 2\frac{q_1}{e}(1+\alpha_2) \pm 2q_2/e < \alpha_2+1 \quad (6)$$

$$\Rightarrow \left|q_1 + \frac{q_2}{1+\alpha_2}\right| < \frac{e}{2} \quad (7)$$

$$E(0,0) < E(0,\pm 1) \Rightarrow [\;]_{(4)} > 0 \overset{1\leftrightarrow 2}{\Rightarrow} \left|q_2 + \frac{q_1}{1+\alpha_1}\right| < e/2 \quad (8)$$

$$E(0,0) < E(\pm 1,\mp 1) \Rightarrow [\;]_{(5)} > 0 \Rightarrow \frac{|\alpha_2 q_1 - \alpha_1 q_2|}{\alpha_1+\alpha_2} < e/2 \quad (9)$$

CB18

(7), (8) (9) define 6 lines in q_1-q_2 plane, forming a "cropped diamond"

① $q_1 < \frac{e}{2} - \frac{q_2}{1+\alpha_2}$,

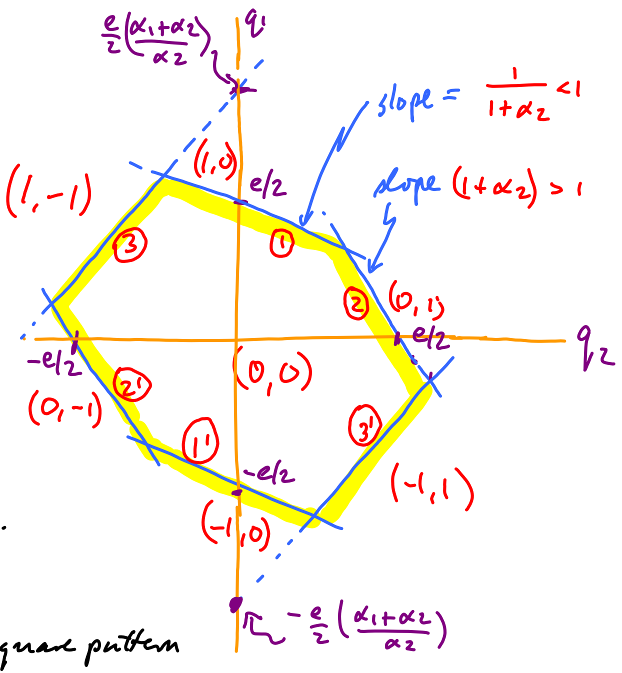
①' $q_1 > -\frac{e}{2} - \frac{q_2}{1+\alpha_2}$

② $q_2 < \frac{e}{2} - \frac{q_1}{1+\alpha_1}$,

②' $q_2 > -\frac{e}{2} - \frac{q}{1+\alpha}$

③ $q_1 < e/2\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}q_2$

③' $q_1 > -e/2\left(\frac{\alpha_1+\alpha_2}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}q_2$



Filling plane gives hexagonal pattern.

- If $\alpha_1, \alpha_2 \gg 1$: $C_{12} \ll C_1^{(g)}, C_2^{(g)}$
  charges N_1, N_2 do not interact ⇒ square pattern
- If $\alpha_1, \alpha_2 \simeq 0$, strong interactions, tiling gives brick wall in (-1,1) direction.

Local Gate Control of a Carbon Nanotube Double Quantum Dot

Science **303**, 655 (2004)

C819

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We have measured carbon nanotube quantum dots with multiple electrostatic gates and used the resulting enhanced control to investigate a nanotube double quantum dot. Transport measurements reveal honeycomb charge stability diagrams as a function of two nearly independent gate voltages. The device can be tuned from weak to strong interdot tunnel-coupling regimes, and the transparency of the leads can be controlled independently. We extract values of energy-level spacings, capacitances, and interaction energies for this system. This ability to control electron interactions in the quantum regime in a molecular conductor is important for applications such as quantum computation.

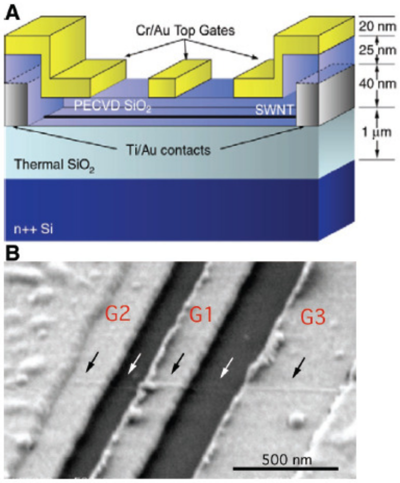

Fig. 1. (**A**) Schematic of top-gated device. (**B**) Electron micrograph of a representative device. Arrows indicate the embedded nanotube.

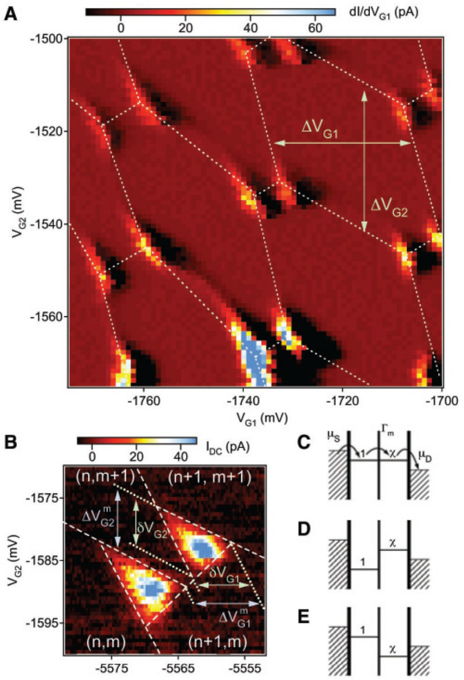

Fig. 2. Experimental charge stability diagrams for the series double quantum dot as a function of two gate voltages, each shifting the energy levels of a single dot. Voltages V_{G1} and V_{G2} are divided by five before being applied to the gates of the device. (**A**) Color scale displays dI/dV_{G1} calculated from dc current (I_{DC}) at V_{SD} = 500 µV. White lines are guides to the eye showing the honeycomb pattern of peaks in conductance. Vertex pairs correspond to points of degeneracy between the two dots where resonant transport occurs, whereas cotunneling may produce finite conductance along the honeycomb edges. (**B**) Zoom-in of a vertex pair at V_{SD} = 500 µV with lines of constant energy designated by white dashed lines. Colorscale displays I_{DC}. Vertex dimensions are indicated on the diagram, as are particular electron configurations on the two dots for different regions of gate space. Note the triangular shape resulting from finite V_{SD}, and nonzero off-resonance conductance within the triangles. (**C**) Level diagrams for a double quantum dot depicting resonant transport with Γ_m the interdot tunnel rate. The bias window is $V_{SD}=(\mu_S-\mu_D)/e$. (**D** and **E**) Level diagrams depicting configurations where cotunneling may contribute to transport.