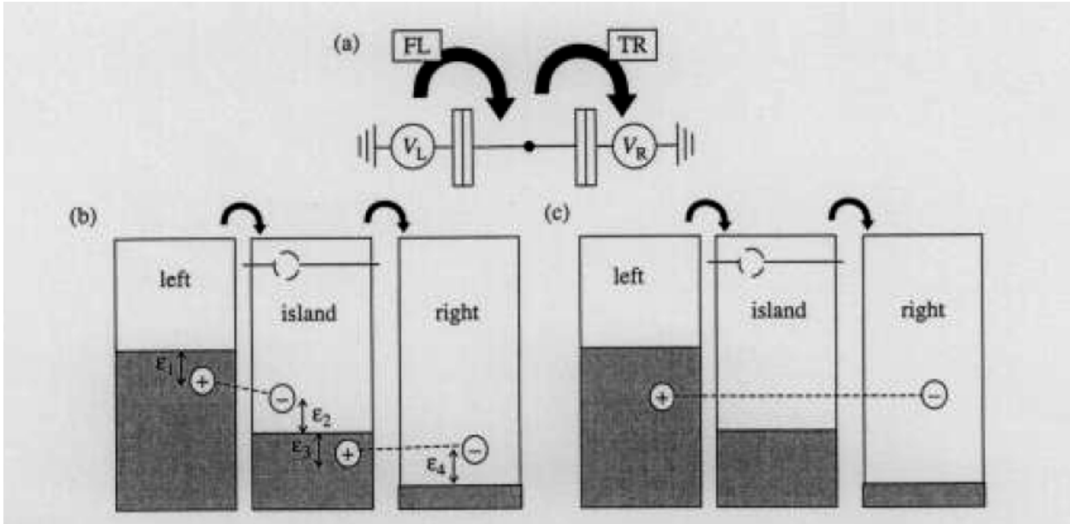
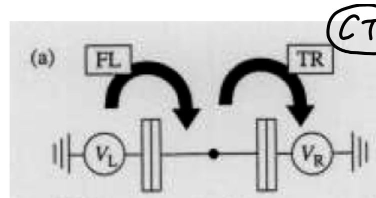
3.4 Cotunneling : Quantum effects in Coulomb blockade

CT1



Co-tunneling in a SET. (a) Co-tunneling: simultaneous transfer of two electrons in two junctions is not forbidden by energy conservation. (b) Energy diagram for inelastic co-tunneling: two electron–hole pairs remain in the final state. (c) Energy diagram for elastic co-tunneling: the transferred electron keeps its energy.

CT2



Rate to cross left junction : Γ_{FL} (1)

Time e can stay on island : $\tau_H \simeq \frac{\hbar}{E_C}$ (2)

Rate to cross right junction : Γ_{TR} (3)

Probability to tunnel to right during time τ_H : $P_{TR} = \tau_H \Gamma_{TR}$ (4)

Total right to get from L to R: $\Gamma_{cot} = \Gamma_{FL} P_{TR} = \Gamma_{FL} \underbrace{\Gamma_{TR} \frac{\hbar}{E_C}}_{= G/G_Q}$ (5)

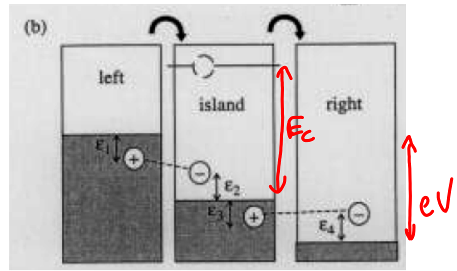
Typical tunneling rates: $\Gamma_{se} \overset{(1.63)}{=} \frac{I}{e} \sim \frac{eVG}{e^2} \sim \frac{E_C G}{e^2} \sim \frac{E_c}{\hbar} \frac{G}{G_Q}$ (6)

$\Rightarrow$ $\Gamma_{cot} = \Gamma_{se} \underbrace{G/G_Q} = \left(\frac{G}{G_Q}\right)^2 \frac{E_C}{\hbar}$ (7)

$\ll 1$ if $G \ll G_Q$ (8)

Same condition (8) ensures Coulomb blockade, so : CB $\Leftrightarrow$ cotunneling is small.

Improve estimate by including energy conservation: (CT3)



Final state has 2 p-h excitations, with energies $\varepsilon_1+\varepsilon_2$ and $\varepsilon_3+\varepsilon_4$. (1)

Without energy conservation, $\varepsilon_i \lesssim E_c$ (2)

With energy conservation: $\sum_{i=1}^{4} \varepsilon_i = eV \Rightarrow \varepsilon_i \lesssim eV$ (3)
(at $k_BT \ll eV$)

# of states available for tunneling is reduced by $\left(\frac{eV}{E_c}\right)^3$ ← 3 of the 4 ε_i are independent, but each $\lesssim eV$. (4)

Previous estimate for Γ_{cot} is reduced by same factor:

$$\Gamma_{cot} \simeq \Gamma_{FL}\Gamma_{TR}\frac{\hbar}{E_c}\left(\frac{eV}{E_c}\right)^3 \quad (5)$$

(2.6) $\frac{\hbar\Gamma_\alpha}{E_c} \simeq \frac{g_\alpha}{g_Q}$

$$\simeq \boxed{\frac{g_L}{g_Q}\frac{g_R}{g_Q}\cdot\left(\frac{eV}{E_c}\right)^2\frac{eV}{\hbar}} \quad (6)$$

At $k_BT \gg eV$, relevant energy window is set by k_BT, (not eV): (CT4)

(3.6) becomes:

$$\boxed{\Gamma_{cot} \overset{(3.6)}{\simeq} \frac{g_L}{g_Q}\frac{g_R}{g_Q}\cdot\left(\frac{k_BT}{E_c}\right)^2\frac{k_BT}{\hbar}} \quad (1)$$

Rate from $L\to R$ and $R\to L$ differ only by small factor eV/k_BT.

$$\Rightarrow \text{Current}: \quad I \simeq e\,\Gamma_{cot}\frac{eV}{k_BT} \simeq \underbrace{e\frac{eV}{\hbar}}_{g_QV}\frac{g_L}{g_Q}\frac{g_R}{g_Q}\cdot\left(\frac{k_BT}{E_c}\right)^2 \quad (2)$$

In general: cotunneling gives <u>non-linear</u> (in V or T) contributions to I-V curve:

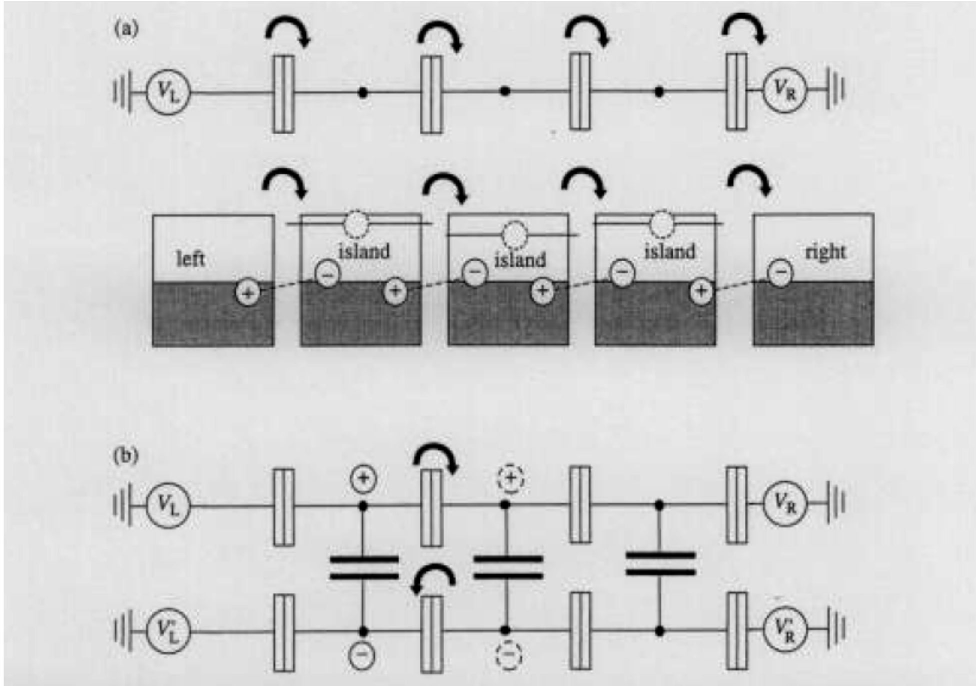
Generalize to N-junction arrays: $\Delta E \simeq \max[k_BT, eV]$

cotunneling yields
$2N$ p-h pairs,
$2N-1$ ε_i's $\simeq \frac{\Delta E}{E_c}$

$$\Gamma_{cot} \overset{(2.7)}{\simeq} \left(g/g_Q\right)^N E_c/\hbar \quad \text{for } \Delta E \gtrsim E_c \quad (3)$$

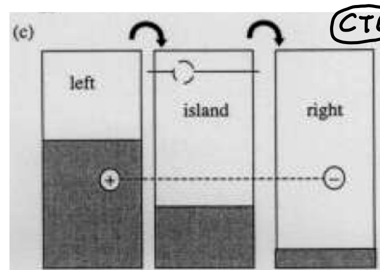
$$\Gamma_{cot} \simeq \left(g/g_Q\right)^N\left(\frac{\Delta E}{E_c}\right)^{2N-\cancel{1}\,2}\frac{\cancel{E_c}}{\hbar}\frac{\Delta E}{\cancel{E_c}} \quad \text{for } \Delta E \lesssim E_c \quad (4)$$

CT5



(a) Complex co-tunneling process: charge transfer in four-junction array. Four electron-hole pairs are left in the final state. (b) Co-tunneling enables transport of artificial excitons – electron-hole pairs in capacitively coupled arrays.

CT6



"Elastic cotunneling": same electron tunnels twice

Total rate for tunneling out to right (from any state): $\Gamma_{se} \overset{(2.6)}{\simeq} \frac{g_R}{g_Q}\frac{E_c}{\hbar}$ (1)

Rate for tunneling from a given state is smaller by a factor δ_s/E_c $\Rightarrow$ $\Gamma_{given} \simeq \frac{g_R}{g_Q}\,\delta_c/\hbar$ (2)

Since same electron tunnels, the energy-conservation penalties $\left(\cancel{\frac{eV}{E_c}}\right)^2$ of (3.4) for intermediate particle hole pair, that usually suppresses inelastic cotunneling are absent:

$$\Gamma_{el\text{-}cot} \overset{(4.1)}{\simeq} \cancel{\frac{\hbar}{e^2}}\,\frac{g_L}{\cancel{g_Q}}\,\frac{g_R}{g_Q}\cdot\cancel{\left(\frac{eV}{E_c}\right)^2}\,\frac{\cancel{e}V}{\cancel{\hbar}}\,\underbrace{\frac{\delta_c}{E_c}}_{\text{from (2)}} = \frac{V}{e}\overbrace{\left(\frac{g_L g_R}{g_Q}\right)\frac{\delta_c}{E_c}}^{g_{el}} = \frac{V}{e}\,g_{el} \quad (3)$$

"elastic cotunneling"

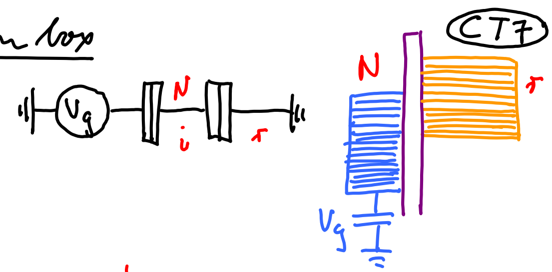
$$\Gamma_{cot} \overset{(3.6)}{\simeq} \overset{(3)}{\Gamma_{el\text{-}cot}} \quad \text{for} \quad \left(\frac{\Delta E}{E_c}\right)^2 \lesssim \frac{\delta_c}{E_c} \Rightarrow \boxed{\Delta E \lesssim \sqrt{\delta_s E_c}} \quad (4)$$

$\Rightarrow$ at sufficiently small excitations, cotunneling is always elastic!!

3.4.1 Quantum effects in single-electron box

CT7

Write $H = H_0 + H_{tun}$



$$H_{tun} \overset{(SET 19.1)}{=} \sum_{ir} \left[T_{ir}\, \hat{a}_i^{\dagger} a_r + T_{ir}^{*}\, a_r^{\dagger} a_i \right] \quad (1)$$

Let $|g\rangle$ = ground state of H_0 (without tunneling) $= \left(\prod_{\varepsilon_r < \varepsilon_F} c_r^{\dagger}\right)_{lead} \left(\prod_{\varepsilon_i < \varepsilon_F} c_i^{\dagger}\right) |Vac\rangle$

$|\tilde{g}\rangle$ = " " of H (with tunneling) (2)

Probability that system described by $|\tilde{g}\rangle$ is not found in $|g\rangle$: $\sim \Gamma_{se}\, \tau_H \sim G/G_0$

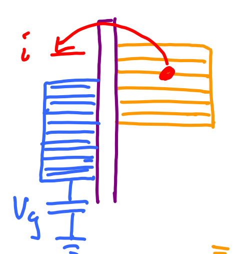
more explicitly: $|\tilde{g}\rangle = |g\rangle + \underbrace{\sum_n \psi_n |n\rangle}$ ← other eigenstates of H_0 (3)

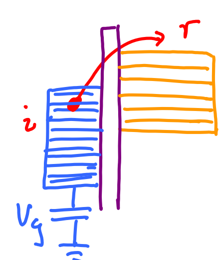
admixtures produced by H_{tun}

1st. order pert. theory in H_{tun}: $\psi_n = \dfrac{\langle n | H_{tun} | g\rangle}{E_n - E_g}$ (4)

CT8

Suppose $|g\rangle$ has $N=0$.





$$\langle n | H_{tun} | g\rangle \overset{(7.1)}{=} \sum_{ri} T_{ir}\; {}_{N=1}\langle n | \overbrace{c_i^{\dagger} c_r | g\rangle}^{\equiv |i,\bar{r}\rangle_{N=1}} + \sum_{ri} T_{ri}^{*}\; {}_{N=-1}\langle n | \overbrace{c_r^{\dagger} c_i | g\rangle}^{\equiv |r,\bar{i}\rangle_{N=-1}} \quad (1)$$

contributes only if, in $|g\rangle$: $\Downarrow$ i = empty, r = filled; $\Downarrow$ r = empty, i = filled

$E_n - E_g = \varepsilon_i - \varepsilon_r + E^{(+)}$ (2) $\qquad E_n - E_g = \varepsilon_r - \varepsilon_i + E^{(-)}$

$E^{(+)} = E_{el}^{(1)} - E_{el}^{(0)} > 0$ (3) $\qquad E^{(-)} = E_{el}^{(-1)} - E_{el}^{(0)}$

$$|\tilde{g}\rangle = |g\rangle_{N=0} + \sum_{i,r}{}' \frac{T_{ir}}{\varepsilon_i - \varepsilon_r + E^{(+)}} |i,\bar{r}\rangle_{N=1} + \sum_{i,r}{}' \frac{T_{ir}^{*}}{\varepsilon_r - \varepsilon_i + E^{(-)}} |r,\bar{i}\rangle_{N=-1} \quad (4)$$

restricted to: (r = filled, i = empty) in $|g\rangle$ $\qquad$ restricted to: (r = empty, i = filled) in $|g\rangle$

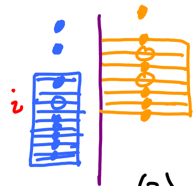
When calculating thermal averages, average over all initial states: CT9

$$\langle g|\hat{O}|g\rangle \longrightarrow \sum_g P_g \langle g|\hat{O}|g\rangle = \sum_g P_g \Big[\langle g|\hat{O}|g\rangle + \sum_n \left(\langle g|O|n\rangle \psi_n + c.c.\right) + \sum_{nn'} \psi_n^* \psi_{n'} \langle n|O|n'\rangle\Big] \quad (1)$$

P_g = Probability that system initially is in state $|g\rangle$

= " " bare system $|g\rangle$

$P_N \equiv$ Probability to have charge N



$$P_1 = \sum_g P_g \langle g|\underbrace{N=1\rangle\langle N=1}_{\text{Projector on } N=1 \text{ component of } |g\rangle}|g\rangle \quad (2)$$

(diagonal: $\langle n|o|n'\rangle \neq 0$ only if $n=n'$) $\Rightarrow r=r',\ i=i'$

$$= \sum_{i,r} \frac{|T_{ir}|^2}{(\varepsilon_i - \varepsilon_r + E^{(+)})^2} \underbrace{f(\varepsilon_r)\left[1-f(\varepsilon_i)\right]}_{\text{Probability that } (r=\text{filled},\ i=\text{empty}) \text{ in } |g\rangle} \quad (3)$$

$$P_{-1} = \sum_g P_g \langle g|N=-1\rangle\langle N=-1|g\rangle \quad (4)$$

$$= \sum_{i,r} \frac{|T_{ir}^*|^2}{(\varepsilon_r - \varepsilon_i + E^{(-)})^2} \underbrace{f(\varepsilon_i)\left[1-f(\varepsilon_r)\right]}_{\text{Probability that } (i=\text{filled},\ r=\text{empty}) \text{ in } |g\rangle} \quad (5)$$

Evaluate $\sum_{i,r}$ sums using SET20, trick 2. CT10

$$\sum_{\ell,r} |T_{\ell r}|^2 f(E_\ell)\left[1-f(E_r)\right]\delta(E_r - E_\ell + \Delta E) \overset{(SET20.1)}{=} \frac{\hbar}{2\pi}\Gamma(\Delta E) \quad (1)$$

$$\overset{(SET17.2)}{=} \frac{g}{2\pi^2 g_Q} \frac{\Delta E}{e^{\Delta E/k_B T} - 1} \overset{T=0}{=} \frac{g}{2\pi^2 g_Q} \begin{cases} 0 & \text{if } \Delta E > 0 \\ |\Delta E| & \text{if } \Delta E < 0 \end{cases} \quad (2)$$

So, rewrite

$$P_1 \overset{(9.3)}{=} \int d\xi \underbrace{\sum_{i,r} |T_{i,r}|^2 f(\varepsilon_r)\left[1-f(\varepsilon_i)\right]\delta(\underset{E_r}{\varepsilon_i} - \underset{E_\ell}{\varepsilon_r} - \xi)}_{\frac{\hbar}{2\pi}\Gamma(-\xi)} \frac{1}{(\underbrace{\cancel{\varepsilon_i - \varepsilon_r}}_{\xi} + E^{(+)})^2} \quad (3)$$

$$P_{-1} \overset{(9.5)}{=} \int d\xi \underbrace{\sum_{i,r} |T_{i,r}|^2 f(\varepsilon_i)\left[1-f(\varepsilon_r)\right]\delta(\underset{E_r}{\varepsilon_r} - \underset{E_\ell}{\varepsilon_i} - \xi)}_{\frac{\hbar}{2\pi}\Gamma(-\xi)} \frac{1}{(\underbrace{\cancel{\varepsilon_r - \varepsilon_i}}_{\xi} + E^{(+)})^2} \quad (4)$$

$$P_{\pm 1} = \frac{\hbar}{2\pi} \int d\xi \frac{\Gamma(-\xi)}{(E^{(\pm)} + \xi)^2} \overset{k_B T \ll E_c}{\underset{(2)}{=}} \frac{g}{2\pi^2 g_Q} \int_0^\infty d\xi \frac{\xi}{(E^{\pm} + \xi)^2} \quad (5)$$

CT11

Integral (10.5) has an "ultraviolet" divergence at $\xi \to \infty$!

Reason: we used an inconvenient, bare basis of uncoupled states.

Quick fix: consider a more physical quantity:

<u>average charge</u> in ground state is well-behaved:

$$\langle N \rangle = \sum_N N \cdot p_N = 1 \cdot p_1 + (0)\, p_0 + (-1) \cdot p_{-1} \quad (1)$$

$$\overset{(10.5)}{=} \frac{g}{2\pi^2 g_\alpha} \int_0^\infty d\xi\, \xi \left[\frac{1}{(E^{(+)} + \xi)^2} - \frac{1}{(E^{(-)} + \xi)^2} \right] \quad (2)$$

$$= \frac{g}{2\pi^2 g_\alpha} \ln \frac{E^{(-)}}{E^{(+)}} \quad (3)$$

$$\left[\int dx \frac{x}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2} \ln(ax+b) \right]$$

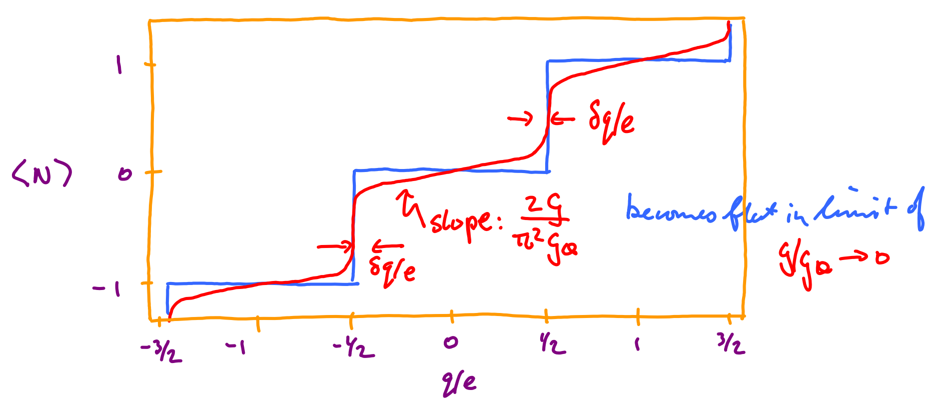
$$\boxed{\langle N \rangle = \frac{g}{2\pi^2 g_\alpha} \ln \left[\frac{1/2 + q/e}{1/2 - q/e} \right]} \quad (4)$$

$$\overset{q/e \to 0}{\simeq} \frac{g}{2\pi^2 g_\alpha} (4\, q/e) \quad (5)$$

(SET 3.1)

$$E^{(\pm)} = E_{el}(\pm 1) - E_{el}(0) = E_c(\pm 1 - q/e)^2 - E_c (q/e)^2 = E_c(1 \mp 2 q/e)$$

CT12



Integral (10.5) $= \int_0^\infty d\xi \frac{\xi}{(E^{(\pm)} + \xi)^2}$ also has infrared divergence (at $\xi = 0$) if $E^{\pm} \to 0$

i.e. if $q/e \to \pm(1/2 - \delta q/e)$. $\Rightarrow$ perturbation theory breaks down when

$$\pm 1 \simeq \frac{g}{2\pi^2 g_\alpha} \ln \left[\frac{1/2 \pm (\frac{1}{2} - \delta q/e)}{1/2 \mp (\frac{1}{2} - \delta q/e)} \right] = \frac{g}{2\pi^2 g_\alpha} \ln \frac{1}{(\delta q/e)^{\pm 1}} \quad (6)$$

$$\Rightarrow \frac{E^{(\pm)}}{2E_c} = \delta q/e \simeq e^{-g/2\pi^2 g_\alpha} \quad (7)$$

Reason: "$|g\rangle$ has $N=0$" no longer holds