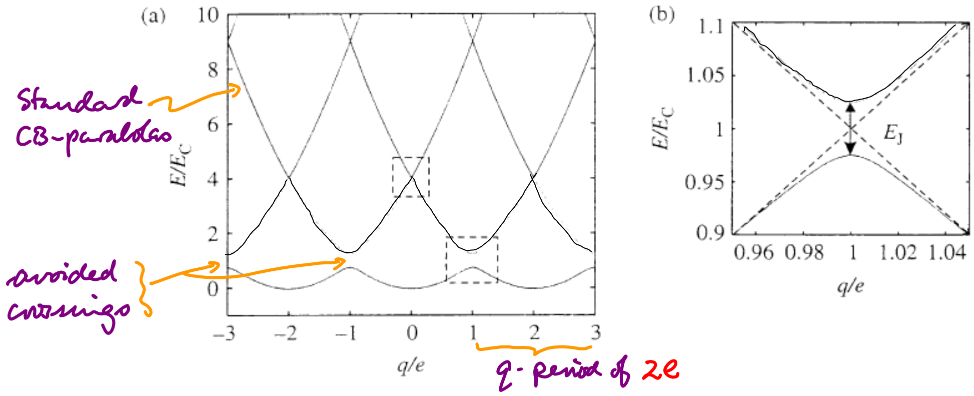
3.5.3 Cooper pair box : $E_c \gg E_J$ "Charging dominates"

MQM17

N is good quantum number, φ fluctuates (1)

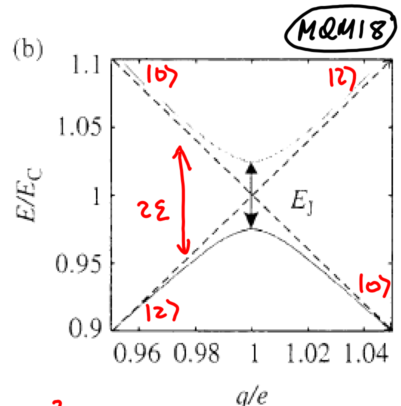
$$H = -\frac{E_J}{2}\left(\hat{\psi}(N+2) + \tilde{\psi}(N-2)\right) + E_c\,(N - q/e)^2$$



(a) Energy levels in a Cooper-pair box at $E_J/E_C = 0.5$ versus q. The quantum states are almost pure charge states except at the parabola crossings. (b) The vicinity of the lowest crossing at $q = 1$ ($E_J/E_C = 0.05$). The eigenstates are superpositions of the two states that cross. A similar picture of repelling levels is valid for any crossing (for example at $q = 0$).

MQM18

Focus on avoided crossing near degeneracy point: $q/e \simeq 1$ (1)



Relevant charge states $|N\rangle$: $|0\rangle, |2\rangle$ (2)

Energies for $E_J = 0$:

$E_c(N) = E_c\,(N - q/e)^2$:

$$E_0 = E_c(q/e)^2$$

$$E_2 = E_c(2 - q/e)^2$$

Energy difference: $E_2 - E_0 = 4E_c(1 - q/e) \equiv 2\varepsilon$ (3)

Effective Hamiltonian in span$\{|0\rangle, |2\rangle\}$:

$$\hat{H} = E_c + \begin{bmatrix} \varepsilon & E_J/2 \\ E_J/2 & -\varepsilon \end{bmatrix} \quad (4)$$

(valid provided $\varepsilon \ll E_c$)

MQM19

Eigenenergies: $E_\pm = E_c \pm \sqrt{\varepsilon^2 + (E_J/2)^2}$ (1)

Eigenstates: $\left.\begin{matrix}|+\rangle \\ |-\rangle\end{matrix}\right\} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 \pm \sin\theta}} \Big(|0\rangle \cos\theta + |2\rangle (\sin\theta \pm 1) \Big)$ (2)

$\theta = \arctan(2\varepsilon/E_J)$ (3)

Normalization: $\langle \pm | \pm \rangle = \frac{1}{2} \frac{1}{1 \pm \sin\theta} \left[\cos^2\theta + \sin^2\theta \pm 2\sin\theta + 1 \right] = 1$ ✓ (4)

Far away from d. point: $\varepsilon/E_J \to \pm\infty$, $\theta \to \pm\pi/2$

upper sign: $|+\rangle \to |2\rangle$, $|-\rangle \to |0\rangle$

lower sign: $|+\rangle \to |0\rangle$, $|-\rangle \to |2\rangle$ (5)

At degeneracy point: $\varepsilon = 0$, $\theta = 0$; $\left.\begin{matrix}|+\rangle \\ |-\rangle\end{matrix}\right\} = \frac{1}{\sqrt{2}} \big(|0\rangle \mp |2\rangle \big)$ (6)

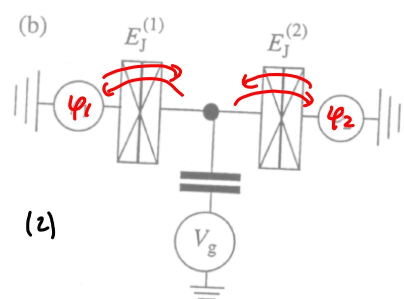
minimum splitting: E_J

superpositions of different charge states!

MQM20

Crossings at higher energies, between $|N\rangle$ and $|N+2m\rangle$ are also avoided, with splitting $E_J (E_J/E_c)^{m-1} \ll E_J$ (1)

Superconducting SSET for $E_c \gg E_J$:



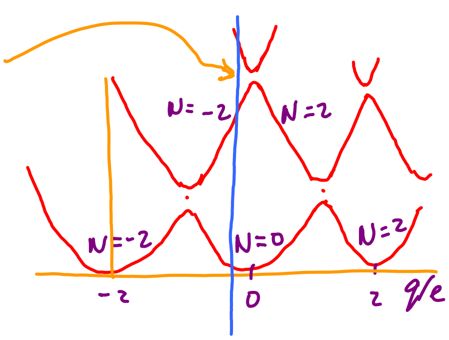
$E_J \to E_J^{\text{eff}}(\varphi_{12}) \overset{(5.4)}{=} \left[E_{J1}^2 + E_{J2}^2 + 2 E_{J1} E_{J2} \cos(\varphi_{12}) \right]^{1/2}$ (2)

$\varphi_{12} = \varphi_1 - \varphi_2$

Consider q/e away from degeneracy point (e.g. near 0):

Virtual tunneling of Cooper pairs from

$|0\rangle \leftrightarrow |2\rangle \leftrightarrow |0\rangle$, $|0\rangle \leftrightarrow |-2\rangle \leftrightarrow |0\rangle$ lowers energy by $E^{(2)}$.



MQM 21

$$E^{(2)} = -\left(\frac{E_J^{(\text{eff})}}{2}\right)^2 \left[\frac{1}{\Delta E_+} + \frac{1}{\Delta E_-}\right] \qquad (1)$$

energy costs of intermediate state $\Delta E_\pm = E_{\pm 2} - E_0$ (2)

Current through SSET (with $E_C \gg E_J$):

MQM 3

$$I_s(\varphi_{12}) = \frac{2e}{\hbar}\partial_{\varphi_{12}} E(\varphi_{12}) \qquad (3)$$

Phase-dependent part of ground state energy (use 20.3)

$$= \frac{2e}{\hbar}\partial_{\varphi_{12}}\frac{1}{4}\left[E_{J1}^2 + E_{J2}^2 + 2E_{J1}E_{J2}\cos(\varphi_{12})\right]\left[\frac{1}{\Delta E_+} + \frac{1}{\Delta E_-}\right] \qquad (4)$$

$$\boxed{I_s = \frac{e}{\hbar} E_{J1}E_{J2}\sin\varphi_{12}\left[\frac{1}{\Delta E_+} + \frac{1}{\Delta E_-}\right]} \qquad (5)$$

3.5.4 Cooper pair box : $E_J \gg E_C$ "Tunneling dominates"

MQM 22

φ is good quantum number, N fluctuates

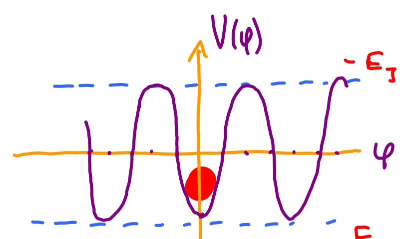
$$m = \frac{\hbar^2}{8E_C}$$

$$H(\varphi) \overset{(12.5)}{=} E_C\Big(-2i\underbrace{\frac{\partial}{\partial\varphi}}_{\equiv\, a\partial_x} - q/e\Big)^2 - E_J\cos\underbrace{\varphi}_{\equiv\, x/a} \qquad (1)$$

"massive particle": $E_C a^2 = \dfrac{\hbar^2}{8m}$

$$m = \frac{\hbar^2}{8E_C}\frac{1}{a^2} \qquad (2)$$

periodic potential.



(3)

small $E_C \Rightarrow$ heavy, almost classical particle

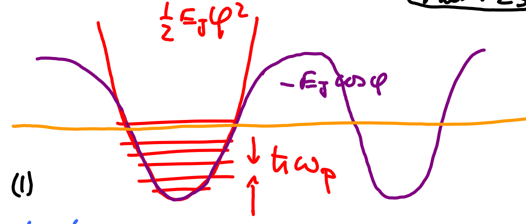
Particle is "trapped" near potential minimum

$\Rightarrow$ phase is fixed

(fluctuations in phase: $\delta\varphi \sim (E_C/E_J)^{1/4} \ll \pi$).

Low-energy excitations near minima: (MQM 23)

Approximate by harmonic oscillator:



$$-E_J\cos\varphi \simeq -E_J + \underbrace{\tfrac{1}{2}E_J}_{\text{"spring constant" } \hat{=}\, k \,=\, E_J}\varphi^2 \qquad (1)$$

$\Rightarrow$ Low-lying states are equidistant, with spacing

$$\underbrace{\hbar\omega_p}_{\text{"plasmon frequency"}} = \hbar\sqrt{\frac{k}{m}} = \hbar\sqrt{E_J\,\frac{8E_C}{\hbar^2}} = \sqrt{8E_JE_C} \ll E_J \qquad (2)$$

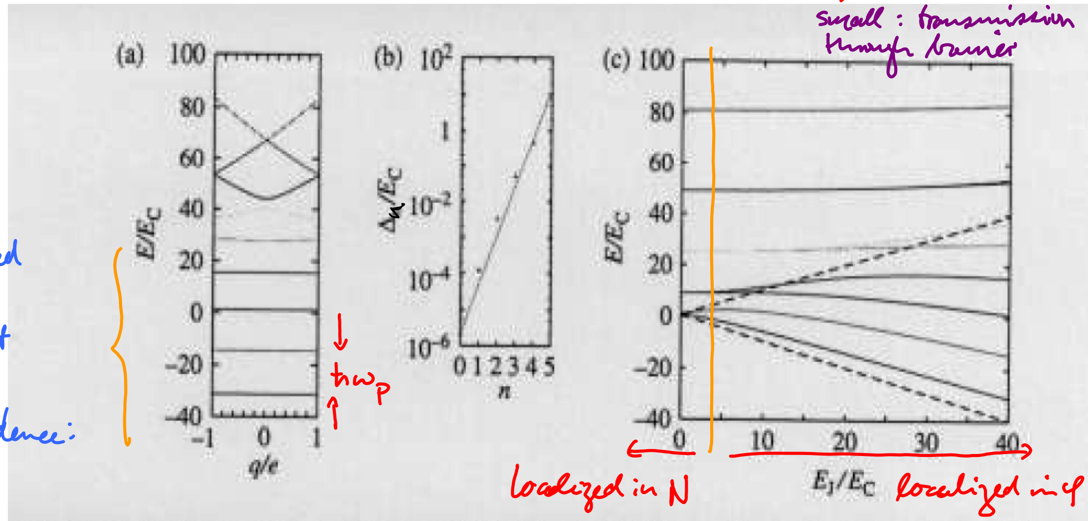
Number of "localized states" (that cannot tunnel through barrier)

can be shown to be $N_c = \left(\frac{8E_J}{\pi^2E_C}\right)^{1/2} \gg 1$ [semiclassical estimate] (3)

q-dependence of eigenenergies is exponentially small, but grows quickly: (MQM 24)

$$E_n(q) = E_n + \Delta_n\cos(\pi q/e), \quad \Delta_n \simeq 16\left(E_J^3E_C/2\pi^2\right)^{1/4}\underbrace{e^{-\pi N_c}}_{\text{small: transmission through barrier}}\cdot e^{\pi n} \qquad (1)$$

N_c localized levels, almost no q-dependence:



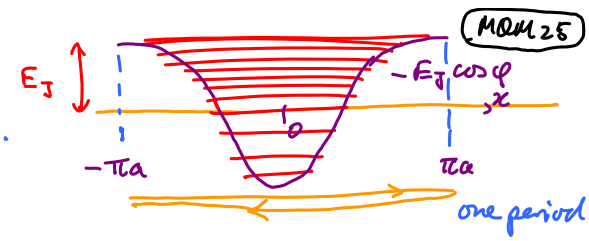
(a) Energy levels in a CPB at $E_J/E_C = 20$ versus q. The states in the energy interval $(-E_J, E_J)$ are localized in phase. (b) The tunneling amplitudes determining the charge dependence of the phase-localized states. The solid line depicts the semiclassical result. (c) Evolution of the states with increasing E_J/E_C at $q = e$.

MQM25

Estimate number of bound states N_c

for particle w. mass $m = \frac{\hbar^2}{8E_c}$ (1)

in potential $V(x) = -E_J \cos x/a$ (2)



Semi-classical Bohr-Sommerfeld quantization:

$$(n+\tfrac{1}{2})h = \oint dx\, p(x) = 4\int_0^{\pi a} dx \sqrt{2m\left[E - V(x)\right]} \qquad (3)$$

integrate over period of motion

At top of barrier: $E = E_J$, $n = N_c$ = # of bound states (4)

$$(N_c + \tfrac{1}{2})h = 4\int_0^{\pi a} dx \sqrt{2m}\cdot\left[E_J\left(1 + \cos x/a\right)\right]^{1/2} \qquad (5)$$

$\gg 1$

$$= 4\sqrt{2mE_J}\left\{2a\sqrt{\cos\tfrac{x}{a} + 1}\tan\tfrac{x}{2a}\right\}_0^{\pi a} \qquad (6)$$

For $x \to a(\pi - \delta)$

$\{\;\} \to 2a\left[\tfrac{1}{2}\delta^2\right]^{1/2}\frac{1}{\delta/2}$

$= 4\sqrt{2}\,a$

$$N_c \simeq 32\, a\frac{\sqrt{mE_J}}{h} = \frac{32a}{2\pi\cancel{\hbar}}\sqrt{E_J}\left[\frac{\cancel{\hbar^2}}{8E_c}\frac{1}{a^2}\right]^{1/2} = \boxed{2\left[\frac{8E_J}{\pi^2 E_c}\right]^{1/2} = N_c} \qquad (7)$$

MQM26

How does q-dependence of Eigenenergies arise?

$$H(\varphi)\,\psi(\varphi) \overset{(12.5)}{=} \left(E_c\left(-2i\frac{\partial}{\partial\varphi} - q/e\right)^2 - E_J\cos\varphi\right)\psi(\varphi) \equiv E\,\psi(\varphi) \qquad (1)$$

It seems: q-dependence can be gauged away:

If $\psi_0(\varphi)$ solves (1) for $q = 0$, (harmonic-oscillator w.f.) (2)

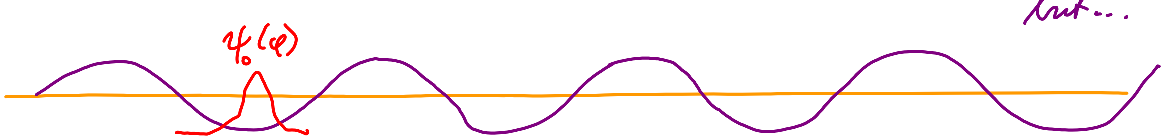
then $\psi_q(\varphi) = \psi_0(\varphi)\, e^{i\frac{q}{2e}\varphi}$ " " " $q \neq 0$. (3)

But: ψ_q is not periodic: $\psi_q(\varphi) \neq \psi_q(\varphi + 2\pi)$? (4)

(see (3.1))

$\uparrow$ for $q \neq 0$

This problem "should not matter" when $\psi_0(\varphi)$ is strongly localized, but...



MQM27

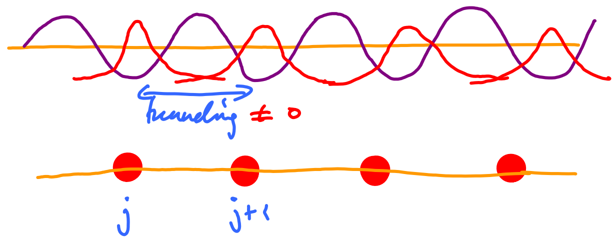
Proper description must involve a superposition of states:

$$\tilde{\psi}(\varphi) = \sum_j a_j \, \psi_0(\varphi + 2\pi j) \qquad (1)$$

harmonic-oscillator wave function for "localized solution"

By construction: $\tilde{\psi}(\varphi) = \tilde{\psi}(\varphi + 2\pi)$ ✓ (2)

If tunneling is small but non-zero, original model can be mapped to tightbinding chain:



$$\langle \varphi | j \rangle \equiv \psi_0(\varphi + 2\pi j)$$

$$H = \sum_j \varepsilon \, |j\rangle\langle j| + \sum_j t \big(|j+1\rangle\langle j| + |j-1\rangle\langle j| \big) \qquad (3)$$

"onsite energy": $\varepsilon = E_0 = \frac{1}{2}\hbar\omega_p$ (4)

"tunneling amplitude": $t = \int d\varphi \, \psi_0^*(\varphi) \, \psi_0(\varphi + 2\pi)$ (5)

MQM28

Solutions of tightbinding chain:

$$|k\rangle = \sum_{j'} e^{ikj'} |j'\rangle \qquad + \sum_j \left(e^{ik(j-1)} |j\rangle + e^{ik(j+1)} |j\rangle \right) \qquad (1)$$

$$H|k\rangle = \varepsilon \sum_j e^{ikj} |j\rangle + \sum_j t \, e^{ikj} \big(|j+1\rangle + |j-1\rangle \big) \qquad (2)$$

$$= \left(\varepsilon + e^{-ik} + e^{ik} \right) \sum_j e^{ikj} |j\rangle \qquad (3)$$

$$= \varepsilon(k) \, |k\rangle \, , \quad \text{with } \varepsilon(k) = \varepsilon + 2t \cos k \qquad (4)$$

Corresponding wave-function:

$$\psi_k(x) \equiv \langle \varphi | k \rangle = \sum_{j'} e^{ikj'} \psi_0(\varphi + 2\pi j) \qquad (5)$$

MQM29

$$\Psi_{q=0}(\varphi) = \langle \varphi | k \rangle = \sum_j \psi_0(\varphi + 2\pi j) \quad \text{corresponds to } k = 0 \tag{1}$$

For $q \neq 0$, we need to gauge-transform:

$$\Psi_{q\neq 0}(\varphi) = \Psi_{q=0}(\varphi)\, e^{i\frac{1}{2}q/e\,\varphi} \tag{2}$$

$$= \sum_j \underbrace{\psi_0(\varphi + 2\pi j)}_{\neq 0 \text{ only for } \varphi \simeq -2\pi j}\, e^{i\frac{1}{2}q/e\,\cancel{\varphi}(-2\pi j)} \tag{3}$$

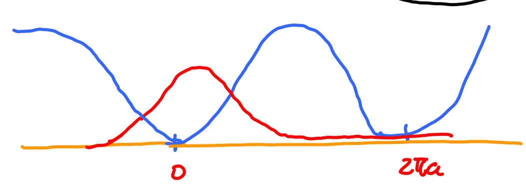
$$= \sum_j \psi_0(\varphi + 2\pi j)\, e^{-i(\pi q/e)j} \tag{4}$$

This is wave-function of the state $|k\rangle$ of (28.5),

with $\boxed{k = -\pi q/e}$, and energy $\boxed{\varepsilon(k) = \varepsilon + 2t \cos \pi q/e}$ (5)

MQM30

Estimate hopping rate:



$$t \overset{(11.5)}{=} \int d\varphi\, \psi_0^*(\varphi)\, \psi_0(\varphi + 2\pi)$$

$$\sim e^{-\frac{1}{\hbar}\int dx\, |p(x)|}$$ WKB-expression for tunneling amplitude.

$$\sim e^{-\frac{1}{\hbar}\underbrace{\int_0^{2\pi a} dx \sqrt{2m E_J}\sqrt{1 - \cos(x/a)}}_{(25.3)\ \equiv\ \frac{1}{2} N_c h}}$$

$$\sim e^{-\pi N_c}$$

Prefactor depends on "attempt frequency"