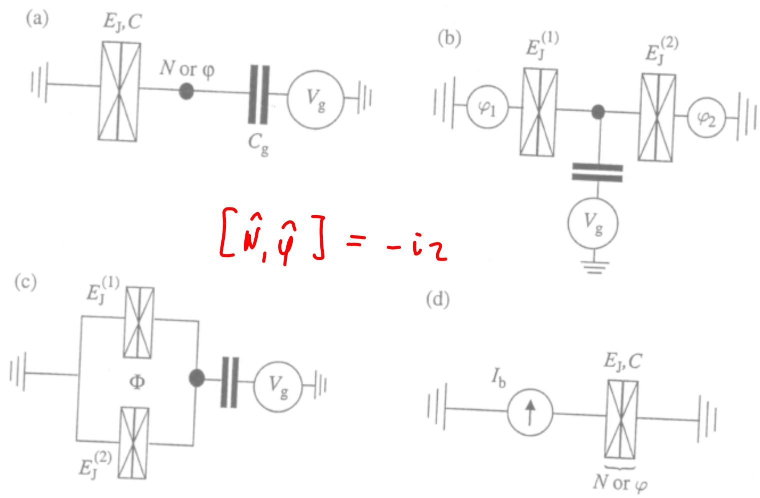
3.5 Macroscopic Quantum Mechanics

MQM1

discrete quantum states from macroscopic systems



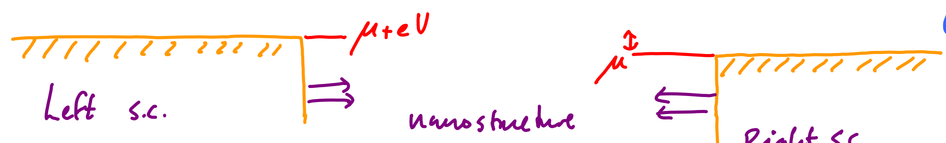
$$[\hat{N}, \hat{\varphi}] = -i_2$$

Cooper-pair box (CPB) and similar systems. A double-crossed box denotes a Josephson junction characterized by the Josephson energy E_J and capacitance C. (a) The CPB. (b) A superconducting SET is equivalent to a CPB with modified E_J. (c) This provides us with an opportunity to tune E_J with magnetic flux Φ. (d) A single current-biased Josephson junction is in many respects similar to a CPB. A crucial difference is that the charge in the capacitor does not have to be discrete.

Reminder about voltage-biased Josephson junction

AR29

MQM2



Phase difference changes: $\dot{\varphi} = \dot{\varphi}_L - \dot{\varphi}_R \overset{(28.5)}{=} 2eV/\hbar$ (1)

Binding energy of all Andreev bound states changes: $\dot{E}_A = \sum_p \dot{E}_p(\varphi) = \sum_p \frac{\partial E_p(\varphi)}{\partial \varphi} \underbrace{\dot{\varphi}}_{2eV/\hbar}$ (2)

Power is dissipated because Super current flows: $IV = P = \dot{E}_g \overset{(24.3)}{=} -\dot{E}_A = -\frac{2eV}{\hbar} \sum_p \frac{\partial E}{\partial \varphi}$ (3)

Super current:
(Flows even for $V \to 0$!)

$$\boxed{I_s = -\frac{2e}{\hbar} \sum_p \frac{\partial E_p(\varphi)}{\partial \varphi} \overset{(24.1)}{=} \frac{\pi \Delta}{2e} G_Q \sum_p \frac{T_p \sin\varphi}{\sqrt{1 - T_p \sin^2 \varphi/2}}} \quad (4)$$

Josephson effect: phase difference $\varphi_L - \varphi_R \neq 0$ induces flow of supercurrent

AR30

MQM3

- Historically, term "Josephson junction" referred to

<u>Tunnel junction:</u> (with $T_p \ll 1$) :

$$I_s(\varphi) = \underbrace{\left(\frac{\pi\Delta}{2e} G_Q \sum_p T_p\right)}_{I_c \,=\, \frac{2e}{\hbar} E_J} \sin\varphi \qquad (1)$$

Critical current:

$$I_c = \frac{\Delta\pi}{2e} \underbrace{G_Q T}_{G_N} = \frac{\Delta\pi}{2e} G_N \qquad (2)$$

conductance of normal junction

Josephson energy:
(defined such that $I_s = \frac{2e}{\hbar}\partial_\varphi E_J(\varphi)$):

$$E_J(\varphi) = -\underbrace{\left(\frac{\hbar}{2e}\right) I_c}_{\equiv E_J} \cos\varphi \qquad (3)$$

Parameter characterizing junction:

$$E_J = \frac{\hbar}{2e} I_c = \frac{\Delta}{4}\frac{G_N}{G_Q} \qquad (4)$$

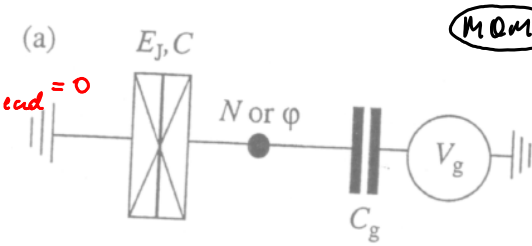
Current:

$$I_s(\varphi) = 2e\,(E_J/\hbar)\sin\varphi \qquad (5)$$

3.5.1 <u>Cooper-pair box</u>

MQM4

(<u>voltage</u>-biased Josephson junction)



Josephson energy of island:

$$E_J(\varphi) \overset{(3.3)}{=} -E_J\cos\varphi \qquad (1)$$

$$E_J \overset{(3.4)}{=} \frac{G_T}{G_Q}\frac{\Delta}{4} \qquad \left(G_N/G_Q \ll 1 \Rightarrow E_J \ll \Delta\right) \qquad (2)$$

Current through junction:

$$I_s(\varphi) \overset{(3.5)}{=} \frac{2e E_J}{\hbar}\sin\varphi \qquad (3)$$

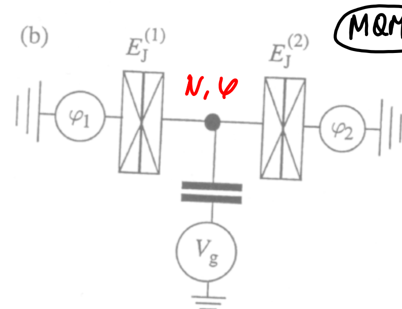
Charging energy of island:

$$E_c(N) = \tilde{E}_c\,(N - q/e)^2 \qquad (\text{important if } G_N/G_Q \ll 1) \qquad (4)$$

$$\tilde{E}_c = \frac{e^2}{2(C+C_g)} \qquad q = C_g V_g \qquad (5)$$

Superconducting SET (SSET)

MQM5



Josephson energy of two junctions:

$$-E_{J1}\cos(\varphi-\varphi_1) - E_{J2}\cos(\varphi-\varphi_2) \qquad (1)$$

$$= -\mathrm{Re}\Big[\underbrace{E_{J1}e^{-i\varphi_1}}_{\hat{E}_{J1}}e^{i\varphi} + \underbrace{E_{J2}e^{-i\varphi_2}}_{\hat{E}_{J2}}e^{i\varphi}\Big] = -\mathrm{Re}\Big[\underbrace{(\hat{E}_{J1}+\hat{E}_{J2})}_{\equiv E_J^{(eff)}e^{-i\tilde{\varphi}}}e^{i\varphi}\Big] \qquad (2)$$

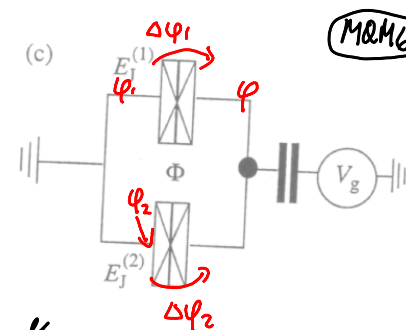
$$= -E_J^{(eff)}\cos(\varphi-\tilde{\varphi}) \qquad (3)$$

$$E_J^{eff} = |\hat{E}_{J1} + \hat{E}_{J2}| = \left[E_{J1}^2 + E_{J2}^2 + 2E_{J1}E_{J2}\cos(\varphi_1-\varphi_2)\right]^{1/2} \qquad (4)$$

E_J^{eff} is tunable if $\varphi_1 - \varphi_2$ can be controlled!

MQM6

applied flux causes phase in superconductor to be non-uniform:



AS31

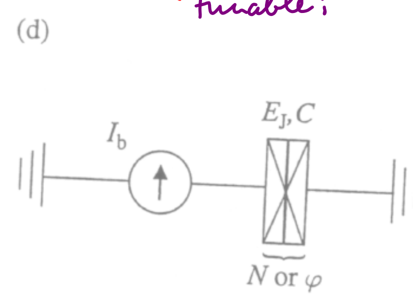
$$2\pi\Phi/\Phi_0 = \Delta\varphi_2 - \Delta\varphi_1 = (\varphi_2-\varphi) - (\varphi_1-\varphi)$$
$$= \varphi_2 - \varphi_1$$

Effective Josephson energy of both junctions together:

$$E_J^{eff} = |\hat{E}_{J1} + \hat{E}_{J2}| = \left[E_{J1}^2 + E_{J2}^2 + 2E_{J1}E_{J2}\cos(2\pi\Phi/\Phi_0)\right]^{1/2}$$

tunable!

Current-biased Josephson junction:



$\varphi = \varphi(t)$ with non-trivial dynamics governed by E_J, E_c.

3.5.2 Second Quantization

MQM7

0) Find classical dynamics of system; to quantize:

i) Replace classical values by operators

ii) Postulate commutation relations between these operators [requires some guesswork!]

iii) Derive Heisenberg eq. of motion from comm. relations

iv) Check if eq. of motion correspond to classical ones.

$$[\hat{A},\hat{B}^2] = \hat{A}\hat{B}^2 - \hat{B}^2\hat{A} = (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{B} - \hat{B}(\hat{B}\hat{A} - \hat{A}\hat{B}) = [\hat{A},\hat{B}]\hat{B} + \hat{B}[\hat{A},\hat{B}] \quad (5)$$

i) $x \to \hat{x}$, $p \to \hat{p}$ (1)

ii) $[\hat{x},\hat{p}] \equiv \alpha$ (not yet known) (2)

iii) $\frac{d}{dt}\hat{x} = -\frac{i}{\hbar}[\hat{x},\hat{H}] = -\frac{i}{\hbar}\left[\hat{x}, \frac{\hat{p}^2}{2m}\right] = \frac{-i}{\hbar}\frac{1}{2m}\left([\hat{x},\hat{p}]\hat{p} + \hat{p}[\hat{x},\hat{p}]\right)$

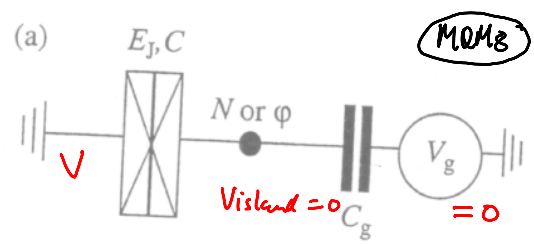
$= \frac{-i}{\hbar m}\alpha\hat{p}$ (3)

iv) $\frac{d}{dt}\hat{x} \overset{\text{check}}{=} \hat{v} = \frac{\hat{p}}{m}$? $\Rightarrow$ $\alpha = i\hbar$ ✓ (4)

MQM8

2nd quantization for Cooper pair box:

0) Find classical dynamics:



$$E_c(N) = \frac{e^2}{2(C+C_g)}(N - q/e)^2 \quad (1)$$

$$E_J(\varphi) \overset{(4.1)}{=} -E_J\cos\varphi \quad (2)$$

$$I_s(\varphi) \overset{(4.3)}{=} I_c\sin\varphi \quad (3)$$

$$eN = Q = CV \quad (8)$$

To simplify discussion, consider:

- $q = 0$, $C_g \ll C$, $\Longrightarrow$ $E_c(N) \overset{(1)}{=} E_c N^2$, $E_c \overset{(4.5)}{=} \frac{e^2}{2C}$ (4)
- $\varphi \ll \pi \Rightarrow E_J(\varphi) \overset{(3)}{\simeq} \text{const.} + \frac{E_J}{2}\varphi^2$ (5)
- $I_s(\varphi) \overset{(2)}{\simeq} I_c\varphi \Rightarrow \dot{I}_s(\varphi) = I_c\dot{\varphi} \equiv V/L$ (6) [(2.1): $\dot{\varphi} = 2eV/\hbar$]
- Define inductance: $L = \frac{\hbar}{2eI_c}$ (7)

Classical equations of motion: MQM 9

Charge conservation: $\frac{1}{e}\frac{dQ}{dt} = -\frac{I(\varphi)}{e} = -\frac{I_c}{e}\varphi = -\frac{2E_J}{\hbar}\varphi$ (1)

Josephson relation: $\frac{d\varphi}{dt} = \frac{2eU}{\hbar} \overset{(8.8)\ Q = CV}{=} \frac{2eQ}{C\hbar} = \frac{4E_c N}{\hbar}$ (2)

Change notation: $Q = eN$, $E_c = \frac{e^2}{2C}$, $I_c = \frac{2e}{\hbar}E_J$ (3)

$$\Rightarrow \boxed{\frac{dN}{dt} \overset{(1)}{=} -\frac{2E_J}{\hbar}\varphi, \quad (4a) \qquad \frac{d\varphi}{dt} = \frac{4E_c}{\hbar}N} \quad (4b)$$

i) Replace classical values by operators: $N \to \hat{N}$ $\quad \varphi \to \hat{\varphi}$ (5)

ii) Postulate commutation relations: $[\hat{N}, \hat{\varphi}] = \alpha$ (6)

$\uparrow$ to be determined

Simplified Hamiltonian: MQM 10

$$\hat{H} = E_c(\hat{N}) + E_J(\hat{\varphi}) = \overset{(8.5)}{\frac{E_J}{2}\hat{\varphi}^2} + \overset{(8.6)}{E_c\hat{N}^2} \quad (1)$$

(iii) Heisenberg eq. of motion: $\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}[\hat{A}, \hat{H}]$ (2)

$$\hat{A} = \hat{\varphi}: \quad \frac{d\hat{\varphi}}{dt} = -\frac{i}{\hbar}E_c\underbrace{[\hat{\varphi}, \hat{N}^2]}_{} = +\frac{i}{\hbar}E_c 2\alpha\hat{N} \overset{(9.4b)}{\equiv} \frac{4E_c\hat{N}}{\hbar} \quad (3)$$

$$(7.5): \underbrace{[\hat{\varphi},\hat{N}]}_{(9.6)\ -\alpha}\hat{N} + \hat{N}\underbrace{[\hat{\varphi},\hat{N}]}_{-\alpha}$$

$\uparrow$ expected from classical eq. of motion $\downarrow$

$$\hat{A} = \hat{N}: \quad \frac{d\hat{N}}{dt} = -\frac{i}{\hbar}\frac{E_J}{2}\underbrace{[\hat{N}, \hat{\varphi}^2]}_{2\alpha\hat{\varphi}} = -\frac{iE_J\alpha}{\hbar}\hat{\varphi} \overset{(9.4a)}{\equiv} \frac{2E_J}{\hbar}\hat{\varphi} \quad (4)$$

iv) Check if eq. of motion correspond to classical ones? $\Rightarrow \alpha = -2i$ (5)

MQM11

(9.6), (10.5) $\Rightarrow$ "$[\hat{N},\hat{\varphi}] = -i2$" (1)

Analogous to. $[\hat{p},\hat{x}] = -i\hbar$ (2) $\Rightarrow$ we can pass over to "first-quantized" description, using wave-functions and Schrödinger equation, by analogy to usual wave mechanics

"Position" $\hat{x}$: $\rightarrow \hat{\varphi}$ (3)

"Wave function" $\psi(x) \equiv \langle x|\Psi\rangle$: $\psi(\varphi) \equiv \langle\varphi|\Psi\rangle$ (4)

"momentum" $\hat{p} \to -i\hbar\frac{\partial}{\partial x}$: $\hat{N} \to -2i\frac{\partial}{\partial\varphi}$ (5)

$$\left[\text{then } [N,\varphi]\psi(\varphi) = -2i\left(\frac{\partial}{\partial\varphi}\varphi\,\psi(\varphi) - \varphi\frac{\partial}{\partial\varphi}\psi\right) = -2i\,\psi(\varphi) \;\checkmark\right] \quad (6)$$

Position- or Phase-representation

MQM12

Hamiltonian (original form): $\hat{H} \overset{(8.1)}{=} E_c(\hat{N} - q/e) - E_J \cos\hat{\varphi}$ (8.2) (1)

Time-independent Schrödinger equation: $\langle\varphi|E|\Psi\rangle = \langle\varphi|\hat{H}|\Psi\rangle$ (2)

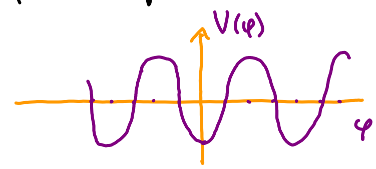
phase-representation: $\psi(\varphi) \equiv \langle\varphi|\Psi\rangle$, $\hat{N} \overset{(11.5)}{\to} -2i\frac{\partial}{\partial\varphi}$: (3)

$$\boxed{E\,\psi(\varphi) = \Big[\underbrace{E_c\left(-2i\frac{\partial}{\partial\varphi} - q/e\right)^2 - E_J\cos\varphi}_{\equiv H(\varphi)}\Big]\psi(\varphi)} \quad (5)$$

"massive particle" $\frac{\hbar^2}{2m} \equiv 4E_c$ (6)

small $E_c \Rightarrow$ heavier, "less quantum" particle

periodic potential.



Boundary conditions on $\psi(\varphi)$: MQM13

$\cos\varphi$ and $H(\varphi)$ are invariant under $\varphi \to \varphi + 2\pi$. Two possible implications:

1. **Either** wave-function is periodic: (applicable to Cooper pair box)

$$|\varphi\rangle = |\varphi + 2\pi\rangle, \quad \psi(\varphi) = \psi(\varphi + 2\pi) \qquad (1)$$

Eigenstates for $E_J \to 0$: (cf (2.5))

$$\psi_N(\varphi) = e^{i\varphi N/2}, \quad \text{with } N \in 2\mathbb{Z} \quad \text{(even integer)}$$

$$E \overset{(12.5)}{=} E_c(N - q/e)^2 \qquad \text{charge quantization!}$$

2. **Or** wave-function is quasi-periodic: (applicable to single JJ under current bias)

$$|\varphi\rangle = e^{i\pi\tilde{q}}|\varphi + 2\pi\rangle, \quad \psi(\varphi) = e^{i\pi\tilde{q}}\psi(\varphi + 2\pi) \quad \text{(like Bloch states in crystals)} \qquad (3)$$

"quasi-charge"

Eigenstates for $E_J \to 0$: $\psi_N(\varphi) = e^{i(N+\tilde{q})\varphi/2}$, with $N \in 2\mathbb{Z}$

but effective charge $(N + \tilde{q})$ can be arbitrary (it gives charge on capacitor plate)

Momentum- or charge representation: MQM14

momentum eigenstates: $|\hat{p}\rangle$, has wavefunction $\langle x|p\rangle = e^{ipx/\hbar}$ $\Big\}\Rightarrow$ $\hat{N}$-eigenstate: $|N\rangle$ (1)

$$\langle\varphi|N\rangle = e^{i\varphi N/2} \qquad (2)$$

Word of caution: literal use of (12.1) leads to problems:

$$\langle N|\hat{N}\hat{\varphi}|N\rangle - \langle N|\hat{\varphi}\hat{N}|N\rangle = \langle N|-i2|N\rangle \qquad (3)$$

$$0 = N\langle N|\hat{\varphi}|N\rangle - \langle N|\hat{\varphi}|N\rangle = -2i \neq 0 \quad !?! \qquad (4)$$

Reason for problem is a theorem from QM:

If $\hat{X}$ ($\hat{N}$) and $\hat{Y}$ ($\hat{\varphi}/2$) are conjugate operators (i.e. $[\hat{X}, i\hat{Y}] = 1$) and the spectrum of $\hat{X}$ are the discrete integers, then $\hat{X}$ is Hermitian only in the space of states "periodic in $\hat{Y}$", i.e. obtained by acting on a reference state (say $|0\rangle$) with *periodic* functions of $\hat{Y}$, i.e. functions depending *only* on the exponentials $e^{\pm i\hat{Y}}$. In

$\langle N|\hat{N}\hat{\varphi}|N\rangle \neq N\langle N|\hat{\varphi}|N\rangle$, since $\hat{\varphi}|N\rangle$ is not "periodic" in $\hat{\varphi}/2$

Remedy: consider only states of the form: $e^{\pm i\hat{\varphi}/2}|N\rangle$ (5)

Then: $\quad e^{\pm i\hat\varphi/2}\,|N\rangle \;=\; ?$ (1) MQM15

Theorem: If $[A,B] = c$, with $[\hat A, c] = 0 = [\hat B, c]$, then $[\hat A, e^{\hat B}] = c\, e^{\hat B}$ (2)

$$\Rightarrow \left[\hat N, e^{\pm i\hat\varphi/2}\right] = \pm i \underbrace{[\hat N, \hat\varphi/2]}_{\overset{(11.1)}{=} -i} e^{\pm i \hat\varphi/2} \qquad (3)$$

view this as a "mnemonic"

$$\boxed{\left[\hat N, e^{\pm i\hat\varphi/2}\right] = \pm\, e^{\pm i\hat\varphi/2}} \qquad (4)$$

{ this relation is well-defined and free of contradictions

$$\Rightarrow \hat N e^{\pm i\hat\varphi/2}|N\rangle = e^{\pm i\hat\varphi/2}\underbrace{(\hat N \pm 1)|N\rangle}_{(N\pm1)|N\rangle} \Rightarrow e^{\pm i\hat\varphi/2}|N\rangle = |N\pm1\rangle \qquad (5)$$

$\Rightarrow$ ladder operator: $\boxed{e^{\pm i\hat\varphi/2\, n} = \sum_N |N\pm n\rangle\langle N|}$ $\forall\, n \in \mathbb{Z}$ (6)

Consistency check: $\langle N|\,\hat N e^{\pm i\hat\varphi/2} - e^{\pm i\hat\varphi/2}\hat N\,|N\rangle \overset{(4)}{=} \langle N|\underbrace{\pm e^{\pm i\hat\varphi/2}|N\rangle}$ (7)

$$0 = N\langle N|e^{(\,)}|N\rangle - \langle N|e^{(\,)}|N\rangle N = \langle N|N\pm1\rangle = 0$$

Hamiltonian in charge representation: MQM16

$$\hat H = E_c(\hat N - q/e)^{\overset{(8.1)}{}} - E_J \underbrace{\cos\hat\varphi}^{(8.2)} \qquad (1)$$

coherent transfer of 2 electrons onto or off island!

$$\tfrac{1}{2}\left(e^{i\hat\varphi} + e^{-i\hat\varphi}\right) \overset{(16.7)}{=} \tfrac{1}{2}\sum_N \left(|N+2\rangle\langle N| + |N-2\rangle\langle N|\right) \qquad (2)$$

Time-independent Schrödinger equation: $\langle N|E|\Psi\rangle = \langle N|\hat H|\Psi\rangle$ (3)

charge-representation: $\tilde\psi(N) = \langle N|\Psi\rangle$, (4)

$$= \int d\varphi\, \underbrace{\langle N|\varphi\rangle}_{\overset{(11.8)}{=} e^{-i\varphi N/2}} \overbrace{\langle\varphi|\Psi\rangle}^{\psi(\varphi)} = \int d\varphi\, e^{-i\varphi N/2}\psi(\varphi) \qquad (5)$$

$$E\tilde\psi(N) = -\frac{E_J}{2}\left(\tilde\psi(N+2) + \tilde\psi(N-2)\right) + E_c\,(N - q/2)^2 \qquad (6)$$