5.3 Quantum Manipulation

QMI

General N-qubit Hamiltonian:
$$\hat{H} = \sum_{i=1}^{N} \hat{H}_i + \sum_{i<j}^{N} \hat{H}_{ij} \tag{1}$$

Single-qubit Hamiltonian:
$$\hat{H}_i = h_a^{(i)} \, \hat{\sigma}_a^{(i)} \tag{2}$$

"x, y or z-handles"

Pauli matrices

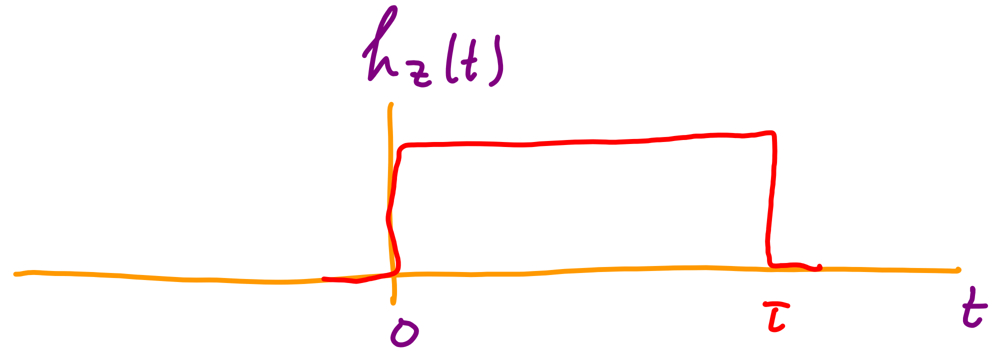
summation implied: $\sum_{a=x,y,z}$

Pairwise interactions:
$$\hat{H}_{ij} = U_{ab}^{(ij)} \, \hat{\sigma}_a^{(i)} \, \hat{\sigma}_b^{(j)} \tag{3}$$

5.3.1 Single-qubit Manipulation

QM2

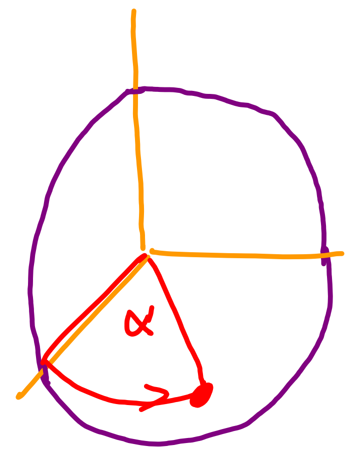
$\hat{H} = h_a \sigma_a$



Let $h_x = h_y = 0$,

Time-evolution:

$$U(\tau) = e^{-iH\tau/\hbar} = e^{-ih_z\sigma_z\tau/\hbar} \quad (1)$$



$$= e^{\frac{i\alpha}{2}\sigma_z}, \qquad \alpha = -\frac{2h_z t}{\hbar} \quad (2)$$

$$= \begin{pmatrix} e^{i\alpha/2} & \\ & e^{-i\alpha/2} \end{pmatrix} = \hat{\mathbb{1}}\cos\frac{\alpha}{2} + \hat{\sigma}_z\, i\sin\frac{\alpha}{2} \quad (3)$$

= rotation about z-axis by angle α.

QM3

If simultaneously $h_x, h_y, h_z \neq 0$:

$$U(\tau) = e^{-i\vec{h}\cdot\vec{\sigma}\,\tau/\hbar} = e^{i\alpha\,\vec{n}\cdot\frac{\vec{\sigma}}{2}} \quad (1)$$

$=$ rotation about axis $\vec{n} = (h_x, h_y, h_z)/h$ (2)

(where $h = \sqrt{h_x^2 + h_y^2 + h_z^2}$) (3)

by angle $\alpha = -2h\tau/\hbar$ (4)

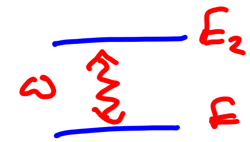
In principle, such "rotation by pulse" is simple.

In practice, problem is: hard to control all three h_a independently. Some may be $\neq 0$ at all times...

Resonant manipulation: ac pulses, rotating wave approximation

QM4

Apply ac-pulse with frequency $\hbar\omega$ (1)



Consider eigenbasis, where $\hat{H}_0 = \hbar\frac{\Omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (2)

near resonance

$$\text{Choose} \quad h_a(t) = \mathrm{Re}\left(2\tilde{h}_a(t)\, e^{-i\omega t}\right), \quad \text{with } \omega \simeq \Omega \tag{3}$$

complex amplitude that varies slowly compared to ω^{-1}

Schrödinger eq:

$$i\hbar\partial_t \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = H \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} = \begin{pmatrix} \frac{\hbar\Omega}{2} + h_z(t) & h_x(t) - i h_y(t) \\ h_x(t) + i h_y(t) & -\frac{\hbar\Omega}{2} - h_z(t) \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \tag{4}$$

If $h_a = 0$, free solution is:

$$\Psi(t) = \begin{pmatrix} \psi_+ \, e^{-i\Omega t/2} \\ \psi_- \, e^{+i\Omega t/2} \end{pmatrix} \tag{5}$$

constants

For $h_\alpha \neq 0$ but $\omega \simeq \Omega$, make Ansatz of similar form: QM5

$$\Psi(t) \equiv \begin{pmatrix} \psi_+(t)\, e^{-i\omega t/2} \\ \psi_-(t)\, e^{i\omega t/2} \end{pmatrix} \qquad (1)$$

"slow" time-dependence on scale of ω^{-1}

$$i\hbar\, \dot{\Psi}(t) = \begin{pmatrix} i\hbar\, \dot{\psi}_+(t) + \hbar\omega/2\, \psi_+(t) \\ i\hbar\, \dot{\psi}_-(t) - \hbar\omega/2\, \psi_-(t) \end{pmatrix} \begin{matrix} e^{-i\omega t/2} \\ e^{+i\omega t/2} \end{matrix} \qquad (2)$$

$$= H\Psi = \begin{pmatrix} \frac{\hbar\Omega}{2} + h_z(t) & h_x(t) - i h_y(t) \\ h_x(t) + i h_y(t) & -\frac{\hbar\Omega}{2} - h_z(t) \end{pmatrix} \begin{pmatrix} \psi_+(t)\, e^{-i\omega t/2} \\ \psi_-(t)\, e^{i\omega t/2} \end{pmatrix} \qquad (3)$$

$(3) \times \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$:

$$i\hbar \begin{pmatrix} \dot{\psi}_+ \\ \dot{\psi}_- \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2}(\Omega - \omega) + h_z(t) & \left(h_x(t) - i h_y(t)\right) e^{i\omega t} \\ \left(h_x(t) + i h_y(t)\right) e^{-i\omega t} & -\frac{\hbar}{2}(\Omega - \omega) - h_z(t) \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \qquad (4)$$

(this equation is still exact)

Rotating wave approximation, version A (RWA-A)

Assume that $\Psi_{\underline{t}}(t)$ does not change significantly on time scale ω^{-1}.

$\Rightarrow$ average $h_a(t)$ over oscillation period: $\langle e^{\pm i\omega t}\rangle = 0$

$$\langle h_z(t)\rangle = \int_0^{2\pi/\omega} dt \, \mathrm{Re}\left(2\hat{h}_z(t)\, e^{-i\omega t}\right) = \int_0^{2\pi/\omega} dt \left[\tilde{h}_z(t)\, e^{-i\omega t} + \tilde{h}_z^*(t)\, e^{+i\omega t}\right] \simeq 0 \tag{1}$$

time-dependence negligible over one period

$$\langle \underset{\overset{\uparrow}{x,y}}{h_a}(t)\, e^{+i\omega t}\rangle = \int_0^{2\pi/\omega} dt \left[\tilde{h}_a(t)\, \cancel{e^{-i\omega t}} + \tilde{h}_a^*(t)\, e^{-2i\omega t}\right] \simeq \tilde{h}_a(t) \tag{2}$$

Effective Hamiltonian in "rotating wave approximation", version A (RWA-A)

$$\hat{H}_{RWA-A} \overset{(5.4)}{=} \begin{pmatrix} \hbar\, \delta\omega/2 & \hat{h}_x(t) - i\,\tilde{h}_y(t) \\ \tilde{h}_x^*(t) + i\,\hat{h}_y^*(t) & -\frac{\hbar}{2}\,\delta\omega \end{pmatrix}, \tag{3}$$

still Hermitian!

$\delta\omega = \Omega - \omega$ = "frequency mismatch"

Comments: QM7

- $\hat{H}_{RWA-A}$ is independent of $\tilde{h}_z(t)$

- If $\tilde{h}_x$, $\tilde{h}_y$ are time-independent, $\hat{H}_{RWA-A}$ is stationary, with eigenvalues determined by

$$(\hbar\,\delta\omega/2 - E)(-\hbar\,\delta\omega/2 - E) = |h_x|^2 + |h_y|^2 \tag{1}$$

"Rabi frequency"

$$\pm\frac{\hbar}{2}\omega_R = E_\pm = \left[\frac{\hbar^2\delta\omega^2}{4} + |\tilde{h}_x|^2 + |\tilde{h}_y|^2\right]^{1/2} \tag{2}$$

- ω_R sets scale for time-dependence of $\psi_\pm(t)$. So, we need:

$$\omega_R \ll \omega, \Omega \tag{3}$$

$\Rightarrow$ limits of applicability of RWA-A: small frequency mismatch: $|\delta\omega| \ll \omega, \Omega$

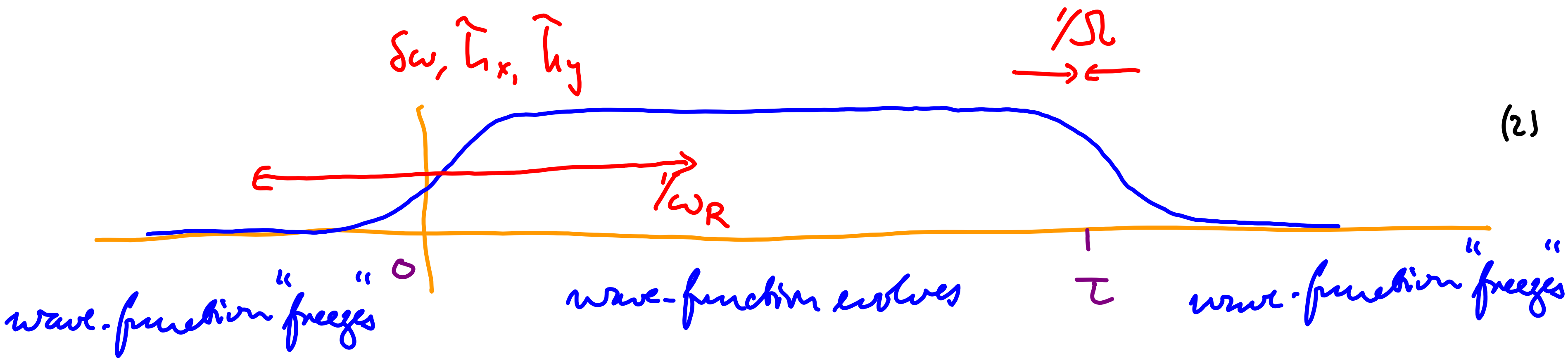
small oscillation amplitudes: $|\tilde{h}_x|, |\tilde{h}_y| \ll \omega, \Omega$

QM8

If $\delta\omega$, $\tilde{h}_x$, $\hat{h}_y = 0$, $\Rightarrow$ $\hat{H}_{RWA} = 0$ (1)

$\Rightarrow$ Wave function does not time-evolve any more

$\Rightarrow$ useful for quantum manipulation! Use pulses that are sharp at slow scale $1/\omega_R$, but smooth at fast scale $1/\Omega$



(2)

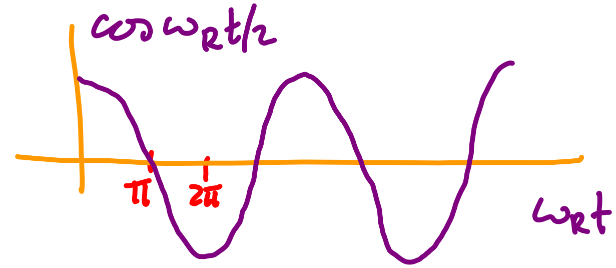
Although $\langle h_z \rangle = 0$, an effective z-handle is $\delta\omega$:

rotation around z-axis can be achieved by $\delta\omega \neq 0$, see (6.3)

Suppose initial condition is
$$\begin{pmatrix} \psi_+(0) \\ \psi_-(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (1)$$

Set $\delta\omega = 0$. Subsequent time-evolution for pulse such as (8.2):

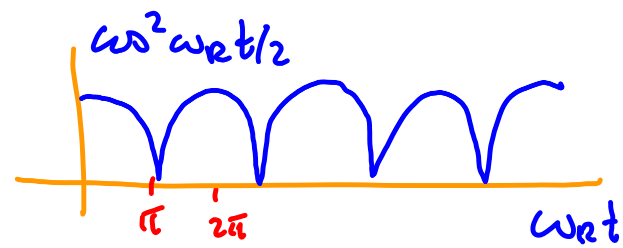
$$\begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \overset{(7.2)}{=} \begin{pmatrix} \cos \omega_R t/2 \\ -i \sin \omega_R t/2 \end{pmatrix} \qquad (2)$$



Probability to remain in initial state:

$$P_+ = |\psi_+(t)|^2 = \cos^2 \omega_R t/2 \qquad (3)$$

= "Rabi oscillations"



On resonance, i.e. for $\omega = \Omega$, Rabi frequency is proportional to amplitude of driving:

$$\frac{\hbar}{2}\omega_R \overset{(7.2)}{=} \sqrt{|\tilde{h}_x|^2 + |\tilde{h}_y|^2} = |\tilde{h}| \qquad (4)$$

Rotating wave approximation, version B (RWA-B)

Useful if there is only a single handle, in basis where it is diagonal:

$$h_z(t) = \tilde{h} \cos\omega t \qquad (1)$$

In this basis, write

$$\hat{H}_0 = \frac{1}{2}\begin{bmatrix} \varepsilon & T \\ T^* & -\varepsilon \end{bmatrix} \quad \text{(need not be diagonal)} \qquad (2)$$

Search for solutions of the form

$$\Psi(t) = \begin{bmatrix} \psi_+(t)\, e^{-i/2\, \phi(t)} \\ \psi_-(t)\, e^{+i/2\, \phi(t)} \end{bmatrix} \qquad (3)$$

Schrödinger eq:

(5.4)

$$i\hbar \begin{pmatrix} \dot\psi_+ \\ \dot\psi_- \end{pmatrix} \overset{(5.6)}{=} \begin{pmatrix} \frac{\hbar}{2}(\Omega-\omega) + h_z(t) & (h_x(t) - i h_y(t))\, e^{i\omega t} \\ (h_x(t) + i h_y(t))\, e^{-i\omega t} & -\frac{\hbar}{2}(\Omega-\omega) - h_z(t) \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \qquad (4)$$

is now replaced by: $\hbar\Omega \to \varepsilon$, $h_x - i h_y \to T$, $\omega t \to \phi$, $\omega \to \dot\phi$ (5)

$$i\hbar \begin{pmatrix} \dot\psi_+ \\ \dot\psi_- \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\varepsilon - \hbar\dot\phi(t)) + h_z(t) & (\frac{1}{2}T)\, e^{i\phi(t)} \\ (\frac{1}{2}T^*)\, e^{-i\phi(t)} & -\frac{1}{2}(\varepsilon - \hbar\dot\phi(t)) - h_z(t) \end{pmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \qquad (6)$$

Now, require that t-dependence of $\phi(t)$ compensates that of $h_z(t)$: OM11

Ansatz: $$\dot{\phi}(t) \equiv \omega + \frac{2 h_z(t)}{\hbar} \overset{(10.1)}{=} \omega + \frac{2\tilde{h}}{\hbar} \cos \omega t \qquad (1)$$

$$\phi(t) = \omega t - \frac{2\tilde{h}}{\hbar\omega} \sin \omega t + \phi_0 \qquad \text{(choose } \phi_0 = 0\text{)} \qquad (2)$$

$$i\hbar \begin{pmatrix} \dot{\psi}_+ \\ \dot{\psi}_- \end{pmatrix} = \begin{bmatrix} \frac{1}{2}(\varepsilon - \hbar\omega) & \frac{1}{2} T e^{i\phi(t)} \\ \frac{1}{2} T^* e^{-i\phi(t)} & -\frac{1}{2}(\underbrace{\varepsilon - \hbar\omega}_{\hbar\delta\omega}) \end{bmatrix} \begin{pmatrix} \psi_+(t) \\ \psi_+(t) \end{pmatrix} \qquad (3)$$

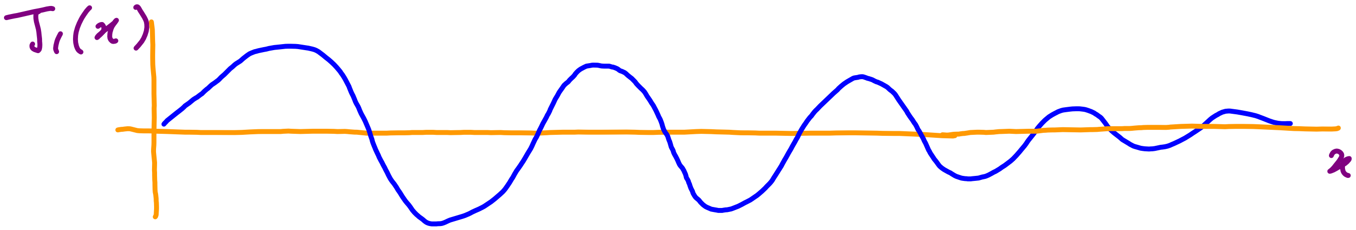
Average over one period: $$H_{RWA\text{-}B} = \frac{1}{2} \begin{bmatrix} \hbar\delta\omega & T\langle e^{i\phi(t)}\rangle \\ T^*\langle e^{-i\phi(t)}\rangle & -\delta\hbar\omega \end{bmatrix} \qquad (4)$$

$$\langle e^{i\phi(t)}\rangle \overset{(2)}{=} \int_0^{2\pi/\omega} dt \; e^{i\omega t} \, e^{-i 2\tilde{h}/\hbar\omega) \sin \omega t} = - J_1(2\tilde{h}/\hbar\omega) \qquad (5)$$

first-order Bessel function

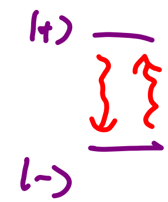
QM12

Off-diagonal elements depend on $\tilde{h}$ in complicated way:



Bessel function gives amplitude of transition between bare states

$|+\rangle \longleftrightarrow |-\rangle$, together with emission or absorption of photon



Eigenenergies of H_{RWA-B} : $\pm\hbar\omega_R = \left[\delta\omega^2/4 + |T|^2 J_1^2(2\tilde{h}/\hbar\omega)\right]^{1/2}$ (1)

Approximation works if $\omega_R \ll \omega, \varepsilon \Rightarrow \delta_{max}\{\delta\omega, |T|\} \ll \omega, \varepsilon$ (2)

Summary:

RWA-A good for : $\begin{cases} \text{handles "perpendicular" to eigenbasis,} & (\tilde{h}_z \text{ not relevant}) \\ \text{manipulation amplitude small,} & |\tilde{h}_x|, |\tilde{h}_y| \ll \omega, \Omega \end{cases}$ (3)

RWA-B good for : $\begin{cases} \text{handles almost parallel to eigenbasis,} & |T| \ll \varepsilon \\ \text{manipulation amplitude } \tilde{h} \text{ can be arbitrary large} \end{cases}$ (4)