Cotunneling Rate

Quantum Mechanical Derivation.

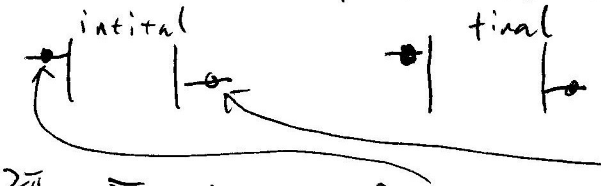
Starting Point: Fermi's Golden Rule:

$$\Gamma_{if} = \frac{2\pi}{\hbar} |\langle i | H_{int} | f \rangle|^2 \delta(E_f - E_i)$$

For single eletron tunnelng:

$$H_{int} = \sum_{l,r} T_{lr} \hat{a}_l \hat{a}_r^{+} + h.c.$$

and te inital / final states are simply



$$\Gamma = \frac{2\pi}{\hbar} \sum_{l,r} |T_{l,r}|^2 f(E_l)(1 - f(E_r)) \delta(E_l - E_r + \Delta E)$$

To convert microscopic $|T_{l,r}|^2$ into observable quantity: insert delta function integral $\int dE_R \delta(E_R - E_r)$

then exchange te sum over l, r with integral $\int dE_L \delta(E_L - E_l)$ and use:

$$\sum_{l,r} |T_{l,r}|^2 \delta(E_r - E_R) \delta(E_L - E_l) = \frac{G}{2\pi^2 G_Q}$$

$$= \frac{\hbar}{2\pi} \Gamma(\Delta E)$$

For cotunnelng, tle intuitive picture of "virtual state" is translated in Q.M. Languge as "Second order perturbatin theory". Specifically, tle Hint has two parts:

$$H_{int} = \sum T_{li} C_l^{+} C_i + h.c. + T_{ir} C_i^{+} C_r + h.c.$$

~~co-tunnel~~i

co-tunneling involve the dot state being modified by one of the tunnely term via <u>second order perturbation</u>

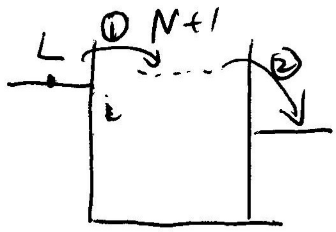
e.g. ~~then~~ ① $\Psi_i = \sum_{n \neq i} \frac{\langle n | T_l C_l C_i^{+} | i \rangle}{E_n - E_i}$

in this case only ~~one~~ states differing in electron number by 1 have no-zero matrix element

then it overlaps via the other ~~matrix element~~ term in Hamiltonian with the other lead:

② $\Gamma = \frac{2\pi}{\hbar} \left| \langle \Psi_i | T_R C_i C_r^{+} | f \rangle \right|^2 \delta(E_l - E \cdots)$

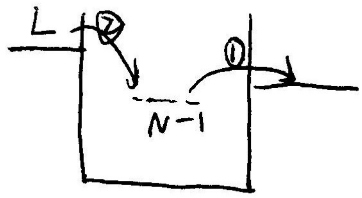
putting it all together, there are two possible processes



Energy denominator is

$E_{v^+} - E_i = E_{el}(N+1) + E_{i_1} - (E_{el}(N) + E_l + eV_L)$

$= \Delta E^+ + E_i - E_l \quad (\Delta E^+ = E_{el}(N+1) - E_{el}(N) - eV_L)$



Energy denominator:

$E_{v^-} - E_i = \Delta E^- + E_r - E_{i_2}$

$\Delta E^- = E_{el}(N-1) - E_{el}(N) + eV_R$

Note: ΔE^- ΔE^+ depend only on physical parameters like charging energies and voltage in the lead

~~E eq.~~ (Sencond equation up from the bottom of PP 255 contains a small <u>typo</u>)

The matrx element in the co-tunnely Ferm's Golden Rule is then

$$M = T_{l i_1} T_{i_2 r} \left(\frac{1}{E^+ + E_{i_1} - E_l} + \frac{1}{E^- + E_r - E_{i_2}} \right)$$

adding in the fermi factors and delta function, we have

$$\Gamma_{cot} = \frac{2\pi}{\hbar} \sum_{l, r, s_1, s_2} |M_{i f i_1 i_2}| \, \delta(E_r - E_l + E_{s_1} - E_{s_2} - eV)$$

$$f(E_l)(1 - f(E_r)) f(E_{s_2})(1 - f(E_{s_1}))$$

top equation on page 256 missed this

~~replace~~ correspondy to i_1, i_2

$$= \frac{2\pi}{\hbar} \int dE_R \, dE_L \underbrace{\Big(\sum_{l, s_1} \delta(E_L - E_l) f(E_l)(1 - f(E_{s_1})) |T_{l i_1}|^2 \Big)}_{\frac{\hbar}{2\pi} \Gamma(E_L - E_{s_1})}$$

$$\underbrace{\Big(\sum_{r, s_2} \delta(E_R - E_r) f(E_{s_2})(1 - f(E_{s_1})) |T_{i_2 R}|^2 \Big)}_{\frac{\hbar}{2\pi} \Gamma(E_{s_2} - E_R)}$$

$$\cdot \left(\frac{1}{E^+ + E_{s_1} - E_l} + \frac{1}{E^- - E_{s_2} + E_r} \right)^2 \delta(E_{s_1} - E_l + E_r - E_{s_2})$$

$\xi_L = E_{s_1} - E_L$ $\qquad \xi_R = E_R - E_{s_2}$

P256 mistake

$$\Gamma^{tot}_{L \to R} = \frac{\hbar}{2\pi} \int d\xi_L \, d\xi_R \, \Gamma_L(-\xi_1) \Gamma_R(\xi_R)$$

$$\left(\frac{1}{E^+ + \xi_L} + \frac{1}{E^- + \xi_R} \right) \delta(\xi_L + \xi_R - eV)$$

where, according to section 3.2.6

$$\Gamma(\Delta E) = \frac{G}{e^2} \frac{\Delta E}{\exp(\Delta E / k_B T) - 1}$$

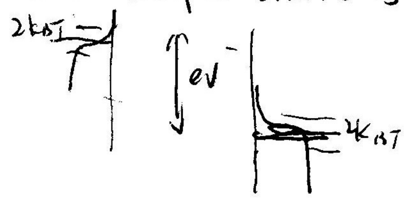
Now, after integration ~~with~~ over ξ_L and ξ_R, one is left with and expression of Γ_{cot} in terms of physical parmeters

$$V \quad \underbrace{V_L \quad V_R \quad \text{and charging energy} \quad \begin{matrix} E_{N+1} - E_N \\ E_{N-1} - E_N \end{matrix}}_{E^+ \; E^-}$$

The rest is algebra.

One simple limit is $eV \gg k_B T$.

P4.



in this case, thermal energy is ~~too~~ small to activate $R \to L$ tunneling,

so $\Gamma(-\cancel{\alpha\xi}) \approx 0$: $\xi_L, \xi_R > 0$

because $\xi \gg k_B T \cancel{\geq 0}$ for most of the tunneling range

$$\Gamma(-\xi_{L,R}) \approx \frac{G^{L,R}}{e^2} \frac{-\xi}{\underbrace{\exp\left(\frac{-\xi}{k_BT}\right)}_{0} - 1} \approx \frac{G^{L,R}}{e^2}\xi$$

$$I_{cot}(V) = e\Gamma_{cot} = \frac{\hbar G_L G_R \cancel{V}}{2\pi\, e^{\cancel{4}}} \int d\xi_L\, d\xi_R\, \xi_L \xi_R \left(\frac{1}{E^+ + \xi_L} + \frac{1}{E^+ + \xi_R}\right)^2$$

$$\delta(\xi_L + \xi_R - eV)$$

the integral for the two square terms are simple, as one can perform the one over delta function trivially:

$$\int d\xi_R \frac{\xi_R}{(E^+ + \xi_R)^2} \int d\xi_L (\delta_{\xi_L} + \xi_R - eV)\, \xi_L$$

$$= \int d\xi_R \frac{\xi_R (eV - \xi_R)}{(E^+ + \xi_R)^2}$$

The cross term is slightly tricker

$$2\int d\xi_L\, d\xi_R \frac{\xi_L \xi_R}{(E^+ + \xi_L)(E^+ + \xi_R)} \delta(\xi_L + \xi_R - eV)$$

$\alpha = \xi_L + \xi_R$, $\beta = \xi_L - \xi_R$

$$\longrightarrow = 2\int d\alpha\, d\beta \frac{\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right)}{\left(E^+ + \frac{\alpha+\beta}{2}\right)\left(E^- + \frac{\alpha-\beta}{2}\right)} \delta(\alpha - eV)$$

single integral

putting it all together, we have

$$I_{cot} = \frac{\hbar G_L G_R V}{2\pi e^2}\left\{\left(1+\frac{2}{eV}\frac{E^+E^-}{E^++E^-+eV}\right)\ln\underbrace{\left[\left(1+\frac{eV}{E^+}\right)\left(1+\frac{eV}{E^-}\right)\right]}-2\right\}$$

Eq. 3.65 missing a bracket!

When bias is small compared with charging energy $E^\pm$, the current ~~conductance is~~ vanishes quadratically:

$$I_{cot}(eV \ll E_c) \;\cancel{\propto \hbar\, V^2 \left(1+ \frac{2E}{\ }\right.}$$

$$\approx \frac{\hbar G_L G_R V}{2\pi e^2}\left[eV\left(1+\frac{E^+E^-}{(E^++E^-)^2}\right)\left(\frac{1}{E^+}+\frac{1}{E^-}\right) + O(eV)^2\right]$$

this is shown in Fig 3.23 (a).

Another limit is $eV \approx k_BT \ll E^{(\pm)}$

in this case, in the interval of relevant ξ, i.e. when $\Gamma(\xi_{L,R})$ is not zero,

$\frac{1}{E^++\xi}$ and $\frac{1}{E^-+\xi}$ is nearly constant, so

we can take the troublesome $\left(\frac{1}{E^++\xi_L}+\frac{1}{E^-+\xi_R}\right)$ out of the integral! The delta function integral is trivial

$$\Gamma^{LR} = \frac{\hbar}{2\pi}\left(\frac{1}{E^+}+\frac{1}{E^-}\right)^2 \int d\xi_L\, \Gamma_L(-\xi_L)\,\Gamma_R(\xi_L - eV)$$

the relevant integral is:

$$\int d\xi_L \frac{-\xi}{\left(\exp(-\xi/k_BT)+1\right)} \frac{(\xi - eV)}{\exp\left(\frac{\xi_L - eV}{k_BT}\right)+1}$$

~~measure~~ take $x = \xi/k_BT$ and $\frac{eV}{2k_BT} = a$

$$-\frac{1}{(k_BT)^2}\exp\left(\frac{eV}{2k_BT}\right)\int dx \frac{x}{e^{(a-x)}+1}\frac{(x-a)}{e^{(x-a)}+1}$$

$$\Gamma^{LR} \propto \int dx \frac{x(x-2a)}{(e^{a-x}+1)(e^{x-a}+1)}$$

$$= \int dx \frac{\frac{a^2}{4} - \left(\frac{x-a}{2}\right)^2}{\cosh^2\left(\frac{x-a}{2}\right)} \qquad \frac{x-a}{2} = z$$

$$= \frac{a^2}{2} \int dz\, [\cosh(z)]^{-2} - \int dz \frac{z^2}{\cosh^2 z}$$

the first integral is trivial;

$$(\tanh x)' = \frac{\cosh^2 - \sinh^2}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$$

so $\int dz\, [\cosh z]^{-2} = \tanh(x)$

The second integral needs to be done by integral by parts twice and using

$$\int dz\, [\cosh z]^{-2} = \tanh(z)$$

$$\int dz\, [\tanh(z)] = \int \frac{d \cosh z}{\cosh z} = \ln[\cosh(z)]$$

putting it all together, one have

$$\Gamma^{LR} = \frac{\hbar G_L G_R}{12\pi e^4} \left(\frac{1}{E^+} + \frac{1}{E^-}\right)^2 \frac{eV((eV)^2 + (2\pi k_B T)^2)}{1 - \exp(-eV/k_B T)}$$

$$\Gamma^{RL} = \Gamma^{LR}(-V)$$

$$I(V) = e(\Gamma^{LR} - \Gamma^{RL}) = \frac{\hbar G_L G_R}{12\pi e^2} \left(\frac{1}{E^+} + \frac{1}{E^-}\right)^2 V[(eV)^2 + (2\pi k_B T)^2]$$

Now let's assuming charging energy E_c dominates $E^\pm$,

The $I(V) \approx \frac{1}{\hbar} \frac{G_L}{G_Q} \frac{G_R}{G_Q} \frac{1}{3\pi^3} eV\left(\left(\frac{eV}{E_c}\right)^2 + \left(\frac{2\pi k_B T}{E_c}\right)^2\right)$

These are precisely (up to numerical factors) Eq. 3.53, 3.54

Finally, similar methods can be used to derive P7

the co-tunneling rate for coherent processes, where the ~~same~~ electron virtually empty an energy level ~~for~~ for cotunneling, in effect creating an electron-hole pair. the rate is expressed as

$$\Gamma_{cot} = \frac{\hbar}{2\pi} \int \prod_{j=1}^{M} \left(\frac{\hbar \Gamma(-\xi_j)\, d\xi_j}{2\pi} \right) S\{\xi_j\} \quad \delta\left(\sum_{j=1}^{M} \xi_j + \Delta E\right)$$

S is the energy denominator, M is the number of electronic transfers. $\{\xi_j\}$ denotes M! possible combinations.