Classification of operators

Let $\hat{A}$, $\hat{B}$ ← will drop hats — be operators, of any of following type:

			sign:	
(i)	b, $b^\dagger$	(bosonic)	$\zeta = +1$	(1)
(ii)	c, $c^\dagger$	(fermionic)	$\zeta = -1$	(2)
(iii)	$c^\dagger_i c_j$, $b^\dagger_i b_j$	(bilinear fermionic or bosonic)	$\zeta = +1$	(3)
(iv)	$c^\dagger_i c^\dagger_j c_\ell$, $c^\dagger_i c_j c_\ell$, ...	(odd # of fermionic op.)	$\zeta = -1$	(4)

$$[A, B]_\zeta \equiv AB - \zeta BA = \begin{cases} \text{commutator for bosons} \\ \text{anti-commutator for fermions} \end{cases} \qquad (5)$$

Example of real-time vs. imaginary time evolution

Quadratic model: (in Schrödinger picture)

$$H_0 = \sum_k \varepsilon_k c^\dagger_k c_k , \qquad [c_k, c^\dagger_{k'}]_\zeta = \delta_{kk'} \qquad (1)$$

ζ ← bosons or fermions

Interaction picture, $t_0 = 0$:

$$c_{k',I}(t) \equiv e^{iH_0 t/\hbar} c_{k'} e^{-iH_0 t/\hbar} = e^{-i\varepsilon_{k'} t/\hbar} c_{k'} \qquad (2)$$

why? $\quad |\Xi\, k'\rangle = c_{k'} |\not{\Xi}\, k'\rangle \qquad (3)$

So:

$$e^{iH_0 t/\hbar} c_{k'} e^{-iH_0 t/\hbar} |\not{\Xi}\, k'\rangle = e^{i(\text{stuff})t} e^{-i(\varepsilon_{k'} + \text{stuff})t} |\Xi\rangle \qquad (4)$$

cancel

$$= e^{-i\varepsilon_{k'} t/\hbar} c_{k'} |\not{\Xi}\, k'\rangle$$

$$\Longrightarrow \quad i\hbar \partial_t c_{k',I}(t) \overset{(2)}{=} \varepsilon_{k'} c_{k',I}(t) \qquad (5)$$

RHS also equals $= [c_{k',I}(t), H_{0,I}(t)] \qquad (6)$

(18.6) is in agreement with general eq. of motion in int. picture: $\boxed{GF19}$

in analogy to (11.2)

$$i\hbar\partial_t \hat{A}_I(t) = [\hat{A}_I(t), \underbrace{\hat{H}_{0,I}(t)}_{=\hat{H}_0 \text{ since } \hat{H}_0 \text{ is } t\text{-indep.}}] + (\partial_t \hat{A})_I(t) \qquad (1)$$

Imaginary time version:

$$\tilde{c}_{k,I}(\tau) = e^{\tau H_0/\hbar} c_k e^{-\tau H_0/\hbar} = e^{-\varepsilon_k \tau/\hbar} c_k \qquad (2)$$

$$-\hbar\partial_\tau \tilde{c}_{k,I}(\tau) = \varepsilon_k \tilde{c}_{k,I}(\tau) \qquad \left[\text{compare } 18.5, \text{ with } \partial_t \to i\partial_\tau\right] \qquad (3)$$

evidently: $\tilde{c}_{k,I}(\tau) = e^{-i\varepsilon_k(-i\tau)/\hbar} c_k \qquad c_{k,I}(t \to -i\tau) \qquad (4)$

illustrating (15.2)

generally: $\tilde{A}(\tau) = A(t \to -i\tau) \qquad (5)$

Thermal Green's function" (TGF) $\boxed{GF20}$

thermal exp. value, (7.1)

Define: $\mathcal{G}_{A,B}(\tau_1;\tau_2) \equiv -\frac{1}{\hbar}\langle T_\tau[\tilde{A}(\tau_1), \tilde{B}(\tau_2)]\rangle \qquad (1)$

T_τ = time-ordering operator along imaginary time:

$$T_\tau[\tilde{A}(\tau_1)\tilde{B}(\tau_2)] = \theta(\tau_1-\tau_2)\tilde{A}(\tau_1)\tilde{B}(\tau_2) + \zeta\,\theta(\tau_2-\tau_1)\tilde{B}(\tau_2)\tilde{A}(\tau_1) \qquad (2)$$

Why this particular definition (1)? Because then single-particle GF have simple eq. of motion:

$$-\hbar\partial_{\tau_1}\mathcal{G}_{c_k c^\dagger_{k'}}(\tau_1,\tau_2) = \delta(\tau_1-\tau_2)\underbrace{[\tilde{c}_k(\tau_1)\tilde{c}^\dagger_{k'}(\tau_2) - \zeta\tilde{c}^\dagger_{k'}(\tau_2)\tilde{c}_k(\tau_1)]}_{(13.5)\equiv[\tilde{c}_k(\tau_1),\tilde{c}^\dagger_{k'}(\tau_2)]_\zeta = \delta_{kk'}} \qquad (3)$$

$$-\frac{1}{\hbar}\langle T_\tau[\overset{(18.3)}{\varepsilon_k}\tilde{c}_k(\tau_1), \tilde{c}_{k'}(\tau_2)]\rangle$$

(18.1), since $\tau_1 = \tau_2$

$$(-\hbar\partial_{\tau_1} - \varepsilon_k)\mathcal{G}_{c_k c_{k'}}(\tau_1,\tau_2) = \delta(\tau_1-\tau_2)\delta_{kk'} \qquad \left(\text{familiar structure for a GF eq. of motion}\right) \qquad (4)$$

Typically, we would solve (20.4) by Fourier transformation. $\boxed{GF21}$
This can be done here, too, but we have to respect:

<u>Periodicity of TGF along τ-axis</u> :

Define $\quad \tau \equiv \tau_1 - \tau_2 \quad , \quad$ with $\quad 0 < \tau \le \beta \quad , \quad -\beta < \tau - \beta \le 0$ (1)

$$\text{then} \quad \hbar\, \mathcal{G}_{AB}(\overset{>0}{\tau}) \overset{(20.1)}{=} - \langle \tilde{A}(\tau_1)\, \tilde{B}(\tau_2) \rangle \tag{2}$$

$$= -\frac{1}{Z} \mathrm{Tr}\Big[e^{-\beta H} \big(e^{\tau_1 \hat{H}/\hbar} A\, e^{-\tau_1 \hat{H}/\hbar} \big) \big(e^{\tau_2 \hat{H}/\hbar} B\, e^{-\tau_2 \hat{H}/\hbar} \big) \Big] \tag{3}$$

commute $e^{\tau_1 H}$ to the back, using cyclic property of trace:

$$= -\frac{1}{Z} \mathrm{Tr}\Big[e^{-\beta \hat{H}}\ \hat{A}\, e^{-(\tau_1 - \tau_2)\hat{H}/\hbar}\ \hat{B}\, e^{+(\tau_1 - \tau_2)\hat{H}/\hbar} \Big] \tag{4}$$

Consider also $\boxed{GF22}$

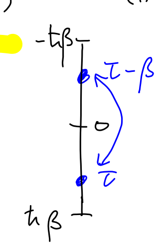
$$\mathcal{G}_{AB}(\overset{<0}{\tau - \beta\hbar}) \underset{(20.1),(20.2)}{=} - \zeta \frac{1}{\hbar} \langle \tilde{B}(\tau_2)\, \tilde{A}(\tau_1 - \beta) \rangle \tag{1}$$

(21.1)

$$= -\zeta \frac{1}{\hbar}\frac{1}{Z} \mathrm{Tr}\Big[\cancel{e^{-\beta \hat{H}}}\ e^{\tau_2 \hat{H}/\hbar} B\, e^{-\tau_2 \hat{H}/\hbar}\, e^{(\frac{\tau_1}{\hbar} - \beta)\hat{H}} A\, e^{-(\frac{\tau_1}{\hbar} - \cancel{\beta})\hat{H}} \Big] \tag{2}$$

$$= -\zeta \frac{1}{\hbar}\frac{1}{Z} \mathrm{Tr}\Big[e^{-\beta \hat{H}}\ \underbrace{e^{\tau_1 \hat{H}/\hbar} A\, e^{-\tau_1 \hat{H}/\hbar}}_{\hat{A}(\tau_1)}\ \underbrace{e^{\tau_2 \hat{H}/\hbar} B\, e^{-\tau_2 \hat{H}/\hbar}}_{\hat{B}(\tau_2)} \Big] \tag{3}$$

$$= -\zeta \frac{1}{\hbar} \mathrm{Tr}\Big[\frac{e^{-\beta \hat{H}}}{Z}\ \hat{A}(\tau_1)\, \hat{B}(\tau_2) \Big] = \zeta\, \mathcal{G}_{AB}(\tau) \tag{4}$$



TGF are periodic/antiperiodic along imaginary time axis
(bosons / fermions)

Matsubara – Transformation

GF 23

discrete Fourier-transform:

periodic for $\zeta = +1$, (bosons) $\quad \omega_n \equiv 2n\pi/(\beta\hbar)$ — Bose frequencies

antiperiodic for $\zeta = -1$ (fermions) $\quad \omega_n \equiv (2n+1)\pi/(\beta\hbar)$ — Fermi frequencies (1)

"Matsubara GF"

Define $\quad \mathcal{G}_{AB}(\tau) \equiv \frac{1}{\beta\hbar} \sum_{n\in\mathbb{Z}} e^{-i\omega_n\tau}\, \bar{\mathcal{G}}_{AB}(i\omega_n)$ (2)

writing i here is a convention. some authors use $\bar{\mathcal{G}}_{AB}(\omega_n)$

Inverse: $\quad \bar{\mathcal{G}}_{AB}(i\omega_n) = \int_0^{\beta\hbar} d\tau\, e^{i\omega_n\tau}\, \mathcal{G}_{AB}(\tau)$ (3)

since $\quad \frac{1}{\beta\hbar}\int_0^{\beta} d\tau\, e^{i(\underbrace{\omega_n-\omega_m}_{\text{always Bose freq.}})} = \begin{cases} 1 & \text{for } n=m \\ e^{2\pi i(n-m)}-1 = 0 & \text{for } n\neq m \end{cases} = \delta_{nm}$ (4)

and $\quad \frac{1}{\beta\hbar}\sum_{n\in\mathbb{Z}} e^{i\omega_n\tau} = \sum_m (-1)^\zeta\, \delta(\tau - m\beta\hbar) =$ periodic/anti-p. δ-function (5)

Obtaining Matsubara GF from eq. of motion

GF 23'

Consider free particles: $\quad H = \sum_k \varepsilon_k c_k^\dagger c_k$ (1)

$$-\partial_\tau c_k(\tau) = [c_k(\tau), H(\tau)] \overset{(1)}{=} \varepsilon_k c_k \quad (2)$$

$$-\partial_\tau \mathcal{G}_{kk'}(\tau) \overset{(20.1)}{=} \delta(\tau)\, \langle \underbrace{\left(c_k(\tau)c_{k'}^\dagger(0) - \zeta c_{k'}^\dagger(0)c_k(\tau)\right)}_{=[c_k,c_{k'}^\dagger]_\zeta = \delta_{kk'}} \rangle$$

$$+ \langle T_\tau \big(\underbrace{[c_k, H]}_{\overset{(2)}{=}\varepsilon_k c_k}\big)(\tau)\, c_{k'}^\dagger(0) \rangle \quad (3)$$

$$(-\partial_\tau - \varepsilon_k)\, \mathcal{G}_{kk'}(\tau) = \delta(\tau)\,\delta_{kk'} \quad (4)$$

Matsubara-transform: $\quad (i\omega_n - \varepsilon_k)\, \bar{\mathcal{G}}_{kk'}(i\omega_n) \overset{\text{(compare 20.4)}}{=} \delta_{kk'}$ (5)

$$\bar{\mathcal{G}}_{kk'}(i\omega_n) = \frac{\delta_{kk'}}{i\omega_n - \varepsilon_k} \quad (6)$$