Complex conjugation:

GF54

$$A^*_{AB}(\omega) \overset{(38.3)}{=} \frac{1}{Z}\sum_{nm} A^*_{nm} B^*_{mn}\left(e^{-\beta E_n} - \xi e^{-\beta E_m}\right)\delta(\hbar\omega - E_m + E_n) = A_{B^\dagger A^\dagger}(\omega) \qquad (1)$$

$$\langle n|A|m\rangle^* \langle m|B|n\rangle^* = \langle n|B^\dagger|m\rangle\langle m|A^\dagger|n\rangle \qquad (2)$$

$$A^*_{AA^\dagger}(\omega) \overset{(1)}{=} A_{AA^\dagger}(\omega) = \text{real} \qquad (3)$$

$$= \frac{1}{Z}\sum_{nm} |A_{nm}|^2 \left(e^{-\beta E_n} - \xi e^{-\beta E_m}\right)\delta(\hbar\omega - E_m + E_n) \qquad (4)$$

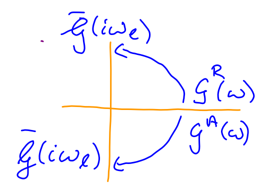
$$= \begin{cases} \geq 0 & \text{for fermions } (\xi = -1) \qquad (5) \\ \geq 0 & \text{for bosons } (\xi = +1) \text{ if } \omega > 0, \text{ since then } E_m > E_n \qquad (6) \end{cases}$$

$$\mathcal{G}^{R*}_{AB}(\omega) = \int d\bar\omega \left(\frac{A_{AB}(\bar\omega)}{\omega + i\alpha - \bar\omega}\right)^* = \int d\bar\omega \, \frac{A_{B^\dagger A^\dagger}(\bar\omega)}{\omega - i\alpha - \bar\omega} = \mathcal{G}^{A}_{B^\dagger A^\dagger}(\omega) \qquad (7)$$

$$\mathcal{G}^*_{AA^\dagger}(\omega) = \mathcal{G}^R_{AA^\dagger}(\omega), \quad (8) \quad \Rightarrow \quad A_{AA^\dagger}(\omega) = \frac{i}{2\pi}\left[\mathcal{G}^R - \mathcal{G}^A\right]_{AA^\dagger}(\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{G}^R_{AA^\dagger}(\omega) \qquad (9)$$

Analytic properties of GF (continued)

GF55



$$\bar{\mathcal{G}}_{AB}(i\omega_\ell) \overset{(42.2)}{=} \int d\bar\omega \, \frac{A_{AB}(\bar\omega)}{i\omega_\ell - \bar\omega}$$

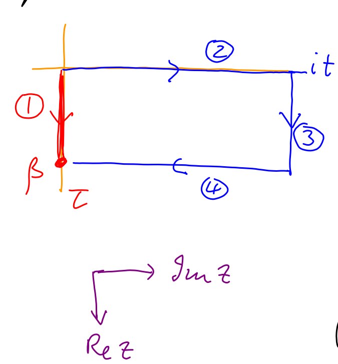
$$\mathcal{G}^{R/A}_{AB}(\omega) = \int d\bar\omega \, \frac{A_{AB}(\bar\omega)}{\omega \pm i\alpha - \bar\omega}$$

$$\Rightarrow \quad \bar{\mathcal{G}}_{AB}(i\omega_\ell) = \mathcal{G}^{R/A}_{AB}(\omega \to i\omega_\ell) \quad \text{for } \omega_\ell \gtrless 0 \qquad (1)\ (2)$$

This relation can also be found directly through contour deformation in time-plane:

$$\bar{\mathcal{G}}_{AB}(i\omega_\ell) \overset{(23.3)}{=} \int_0^\beta d\tau \, e^{i\omega_\ell \tau} \, \underbrace{\mathcal{G}_{AB}(\tau)}_{(36.4)\, =\, G_{AB}(\tau)} \qquad (3)$$

① since $\tau > 0$



For $\omega_\ell > 0$:

Deform contour: ① $\xrightarrow{\text{deform}}$ ② + ③ + ④

① : $\tau \to it$, t from 0 to ∞ (4)

② : $\tau \to i\infty + \tau'$, τ' from 0 to β (5)

$z = \tau + it$

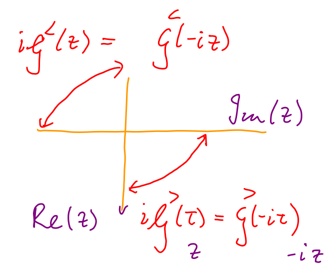
③ : $\tau \to it + \beta$, t from ∞ to 0 (6)

$\boxed{GF56}$

$$\bar{\mathcal{G}}_{AB}(i\omega_\ell) \overset{(55.3)}{=} i\int_{0 \, ②}^{\infty} dt\, e^{-\omega_\ell t}\, \underbrace{\mathcal{G}^{>}_{AB}(it)}_{\equiv z} \; + \; \underset{③}{0} \; + \; i\int_{\infty \, ④}^{0} dt\, e^{-\omega_\ell t}\, \underbrace{e^{+i\omega_\ell \beta}}_{=\xi}\, \mathcal{G}^{>}_{AB}(\underbrace{it+\beta}_{\equiv z}) \qquad (1)$$

$e^{-\omega_\ell \cdot \infty} = 0$

Now recall: $\quad i\mathcal{G}(\tau) = G^{\gtrless}_{AB}(t = -i\tau) \qquad (2)$



So, for general z, with $\mathrm{Re}(z) \geq 0$:

$$i\,\mathcal{G}^{\gtrless}_{AB}(\overset{\tau}{z}) \overset{(2)}{=} G^{\gtrless}_{AB}(-iz) \qquad (3)$$

hence $\quad i\mathcal{G}^{>}_{AB}(it) \overset{(3)}{=} G^{>}_{AB}(t), \qquad (4)$

with $t \in I$ there replaced by $t - i\beta \in I$

$$i\mathcal{G}^{>}_{AB}(it+\beta) \overset{(3)}{=} G^{>}_{AB}(t-i\beta) \overset{(35.4)}{=} \xi G^{<}_{AB}(t) \qquad (5)$$

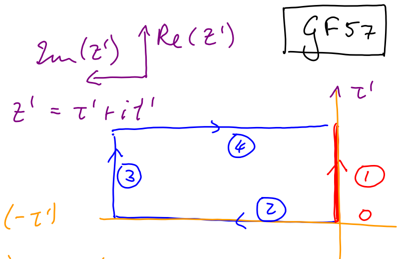
$$\bar{\mathcal{G}}_{AB}(i\omega_\ell) \overset{(1)}{=} \int_{\cancel{0}\,-\infty}^{\infty} dt \left[\overset{(4)}{G^{>}_{AB}(t)} - \overset{(5)}{\xi G^{<}_{AB}(t)} \right] e^{-\omega_\ell t} \qquad (6)$$

$\theta(t)$ can be inserted without changing anything

$$= \int_{-\infty}^{\infty} dt\, \theta(t) \underbrace{\left[G^{>}_{AB}(t) - G^{<}_{AB}(t) \right]}_{(30.4)\; G^R_{AB}(t)} e^{+i(i\omega_\ell t)} \overset{(30.4)}{=} G^R(i\omega_\ell) \quad \text{for } \omega_\ell > 0 \qquad (7)$$

$\boxed{GF57}$

For $\omega_\ell < 0$: $\quad$ set $\tau = -\tau' + \beta$ in (52.3)



$$\bar{\mathcal{G}}_{AB}(i\omega_\ell) \overset{(23.3)}{=} -\int_{\beta}^{0} d\tau'\, e^{i\omega_\ell(-\tau'+\beta)}\, \mathcal{G}_{AB}(-\tau'+\beta) \qquad (1)$$

$\overset{(36.4)}{=} \xi\, \mathcal{G}_{AB}(-\tau') \overset{(36.4)}{=} \mathcal{G}^{<}_{AB}(-\tau')$ since $-\tau' < 0$

$$= \int_{0}^{\beta} d\tau'\, e^{-i\omega_\ell \tau'}\, G^{<}_{AB}(-\tau') \qquad (2)$$

Deform contour: ① $\xrightarrow{\text{deform}}$ ② + ③ + ④

①: $\tau' \to it'$, $\quad t'$ from 0 to ∞

②: $\tau' \to i\infty + \tau'$, $\quad \tau'$ from 0 to β

③: $\tau' \to it' + \beta$, $\quad t'$ from ∞ to 0

$$\bar{\mathcal{G}}_{AB}(i\omega_\ell) = i\int_{0 \, ②}^{\infty} dt'\, e^{+\omega_\ell t}\, \mathcal{G}^{<}_{AB}(-it') \; + \; \underset{③}{0} \; + \; i\int_{\infty \, ④}^{0} dt\, e^{+\omega_\ell t}\, \underbrace{e^{-i\omega_\ell\beta}}_{=\xi}\, \mathcal{G}^{<}_{AB}(\underbrace{-it'-\beta}_{\equiv z}) \qquad (3)$$

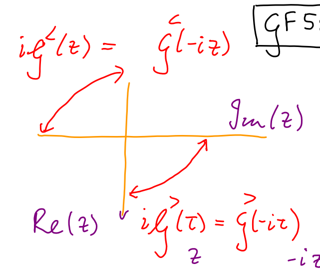
$e^{+\omega_\ell \cdot \infty} = 0$

GF58

Now: $i\,\mathcal{G}^{\gtrless}_{AB}(z) \overset{(56.3)}{=} \mathcal{G}^{\gtrless}_{AB}(-iz)$ for $\mathrm{Re}(z) \geq 0$ (1)

$i\,\mathcal{G}^<_{AB}(-it') \overset{(1)}{=} \mathcal{G}^<_{AB}(-t')$ (2)

$i\,\mathcal{G}^<_{AB}(-it'-\beta) \overset{(1)}{=} \mathcal{G}^<_{AB}(-t'+i\beta) = \zeta\,\mathcal{G}^>_{AB}(-t')$ (3)



$$\bar{\mathcal{G}}_{AB}(i\omega_\ell) \overset{(57.3)}{=} \int_{\cancel{0}\;-\infty}^{\infty} dt' \left[\overset{(2)}{\mathcal{G}^<_{AB}(-t')} - \overset{(3)}{\mathcal{G}^>_{AB}(-t')}\right] e^{\omega_\ell t'} \quad \Theta(t') \quad (4)$$

← can be inserted without changing anything

$t = -t'$

$$= \int_{-\infty}^{\infty} dt' \underbrace{(-)\Theta(-t)\left[\mathcal{G}^>_{AB}(t') - \mathcal{G}^<_{AB}(t')\right]}_{(30.6)\ \mathcal{G}^A_{AB}(t)} e^{i(i\omega_\ell)t} = \mathcal{G}^A_{AB}(i\omega_\ell) \text{ for } \omega_\ell < 0 \quad (5)$$

Shortcut to (58.5):

$$\bar{\mathcal{G}}_{AB}(i\omega_\ell)_{(\omega_\ell<0)} \overset{(42.2)}{=} \int d\bar\omega \frac{A_{AB}(\bar\omega)}{i\omega_\ell - \bar\omega} \overset{(56.1)}{=} \left[\int d\bar\omega \frac{A_{B^\dagger A^\dagger}(\bar\omega)}{-i\omega_\ell - \bar\omega}\right]^* \quad (6)$$

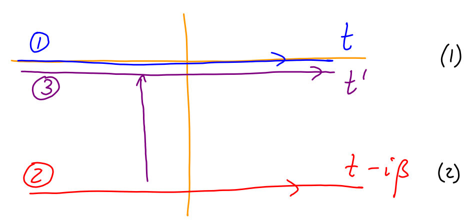
$$= \left[\bar{\mathcal{G}}_{B^\dagger A^\dagger}(-i\omega_\ell)_{-\omega_\ell>0}\right]^* \overset{(56.7)}{=} \left[\mathcal{G}^{R*}_{B^\dagger A^\dagger}\left((-i\omega_\ell)^*\right)\right] \overset{(56.7)}{=} \mathcal{G}^A_{AB}(i\omega_\ell) \quad (7)$$

Expressing $\mathcal{G}^\gtrless$ in terms of spectral function:

GF59

$$\mathcal{G}^<_{AB}(\omega) = \int_{-\infty\ (1)}^{\infty} dt\, e^{i\omega t}\, \mathcal{G}^<_{AB}(t) \quad (1)$$

$$\overset{(35.4)}{=} \zeta \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \underset{(2)}{\mathcal{G}^>_{AB}(t-i\beta)} \quad (2)$$



Deform contour from 2 to 3, so $t - i\beta \to t'$

$$\mathcal{G}^<_{AB}(\omega) = \zeta \int_{-\infty}^{\infty} dt'\, e^{i\omega(t'+i\beta)}\, \mathcal{G}^>_{AB}(t') = \zeta\, e^{-\beta\omega}\, \mathcal{G}^>_{AB}(\omega) \quad (3)$$

Now: $A_{AB}(\omega) \overset{(43.4)}{=} \frac{i}{2\pi}\left[\mathcal{G}^R - \mathcal{G}^A\right]_{AB}(\omega) \overset{(30.5)}{=} \frac{i}{2\pi}\left[\mathcal{G}^> - \mathcal{G}^<\right]_{AB}(\omega)$ (4)

$$\overset{(3)}{=} \frac{i}{2\pi}\left[1 - \zeta e^{-\beta\omega}\right] \mathcal{G}^>_{AB}(\omega) \quad (5)$$

or $\overset{(3)}{=} \frac{i}{2\pi}\left[\zeta e^{+\beta\omega} - 1\right] \mathcal{G}^<_{AB}(\omega)$ (6)

GF60

$$\Longrightarrow \quad G^{\gtrless}_{AB}(\omega) \overset{(59.5,6)}{=} \pm 2\pi i\, \xi \, \frac{1}{e^{\mp\beta\omega} - \xi} A_{AB}(\omega) \tag{1}$$

$$= \pm 2\pi i\, \xi\, n_\xi(\mp\omega)\, A_{AB}(\omega) \tag{2}$$

where $$n_\xi(\omega) = \frac{1}{e^{\beta\omega} - \xi} = \begin{cases} \text{Bose function for } \xi = + \\ \text{Fermi function for } \xi = -1 \end{cases} \tag{3}$$

Check: $$G^> - G^< = (2\pi i)\, \xi \underbrace{\left[\frac{1}{e^{-\beta\omega} - \xi} + \frac{1}{e^{\beta\omega} - \xi}\right]}_{=-1} A_{AB}(\omega) \quad \checkmark \ (43.4) \tag{4}$$

$A_{AB}(\omega)$, $G^{R/A}(\omega)$ encode only "dynamical information":

position of poles determines eigenenergies, decay times, etc. (cf. p. 50)

$G^{\gtrless}_{AB}(\omega)$ also encode "statistical information", via $n_\xi(\mp\omega)$: thermal occupancies

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Fluctuation dissipation theorem

GF61

$$G^>_{AB}(t) \overset{(30.1)}{\equiv} -i/\hbar \, \langle \hat{A}(t)\, \hat{B}(0) \rangle \tag{1}$$

$$G^<_{AB}(t) \overset{(30.2)}{\equiv} -\xi\, i/\hbar \, \langle \hat{B}(0)\, \hat{A}(t) \rangle \tag{2}$$

$$G^<_{AB}(\omega) \overset{(59.3)}{=} \xi\, e^{-\beta\omega}\, G^>_{AB}(\omega) \tag{3}$$

$$\frac{\langle [A,B] \rangle_\omega}{\langle \{A,B\} \rangle_\omega} \equiv \frac{\int dt\, e^{i\omega t} \langle [\hat{A}(t), \hat{B}(0)] \rangle}{\int dt\, e^{i\omega t} \langle \{\hat{A}(t), \hat{B}(0)\} \rangle} = \frac{(G^> - \xi G^<)(\omega)}{(G^> + \xi G^<)(\omega)} \overset{(3)}{=} \frac{1 - e^{-\beta\omega}}{1 + e^{-\beta\omega}} \tag{4}$$

$$= \frac{e^{\beta\omega/2} - e^{-\beta\omega/2}}{e^{\beta\omega/2} + e^{-\beta\omega/2}} = \coth \omega\beta/2 \qquad \text{"fluctuation-dissipation theorem"} \tag{5}$$

"Fluctuations" are characterized by $\langle A(t)\, B(t') + B(t')\, A(t) \rangle = \langle \{A(t), B(t')\} \rangle$

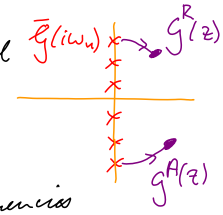
"Dissipation" is characterized by $\langle A(t)\, B(t') - B(t')\, A(t) \rangle = \langle [A(t), B(t')] \rangle$,

because the commutator enters the Kubo formula, which says how a system reacts (dissipates energy) in response to a perturbation.

<u>Analytical continuation from $\bar{\mathcal{G}}$ to $\mathcal{G}^{R/A}$</u>

Suppose $\bar{\mathcal{G}}_{AB}(i\omega_n)$ is known at all $\omega_n = \begin{cases} 2\pi n T \\ 2\pi(n+\tfrac{1}{2})T \end{cases}$ $\forall\, n \in \mathbb{Z}$

To find $\mathcal{G}^{R/A}_{AB}(\omega) = \bar{\mathcal{G}}_{AB}(i\omega_n \to \omega \pm i\alpha)$, we need the analytical continuation of $\bar{\mathcal{G}}$, say $\mathcal{G}(z)$ (with $z \in \mathbb{C}$), such that



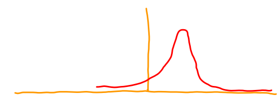
(i) $\mathcal{G}_{AB}(z = i\omega_n) = \bar{\mathcal{G}}_{AB}(i\omega_n)$ at all discrete Matsubara-frequencies

(ii) $\mathcal{G}_{AB}(z)$ is analytic in upper and in lower half-plane

this follows from spectral representation, $\mathcal{G}_{AB}(z) = \int d\bar{\omega}\, \dfrac{A_{AB}(\bar{\omega})}{z - i\bar{\omega}}$ (1)

(iii) $\lim\limits_{|z|\to\infty} z\,\mathcal{G}_{AB}(z) = \text{const.}$ $\left[\text{i.e. } \mathcal{G}(z) \to 0 \text{ at least as fast as } \frac{1}{|z|}\right]$

this follows since $\int d\bar{\omega}\, A_{AB}(\bar{\omega}) \overset{(53.3)}{=} \langle [A,B]_\zeta \rangle = \text{finite}$



$\Longrightarrow$ $A_{AB}(\bar{\omega})$ is integrable function without any weight at $\pm\infty$

$\overset{(1)}{\Longrightarrow}$ $\mathcal{G}_{AB}(z) \sim \frac{1}{z}$ for $|z| \to \infty$

<u>Theorem:</u> Baym and Mermin, (1961):

(i), (ii), (iii) <u>uniquely</u> determine $\mathcal{G}(z)$ in upper and lower half-plane.

Without condition (iii), the analytic continuation $\mathcal{G}(z)$ can <u>not</u> be uniquely determined. Example:

Suppose $\mathcal{G}(z)$ satisfies (i) and (ii), and some function $h(z)$ satisfies (ii)

then the function $\tilde{\mathcal{G}}(z) = \mathcal{G}(z) + \underbrace{(1 - \zeta e^{\beta z})}_{=0 \text{ for } z = i\omega_n} h(z)$ (1)

also satisfies (i) and (ii), i.e. $\mathcal{G}(z)$ is not unique

<u>Example:</u>
$$\mathcal{G}_k(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\, \mathcal{G}_k(\tau) \qquad (2)$$

$\langle cc^\dagger \rangle = 1 + \zeta \langle c^\dagger c \rangle = 1 + \zeta n_\zeta$

$$= -\int_0^\beta d\tau\, e^{i\omega_n\tau} \langle c_k e^{-\varepsilon_k\tau} c_k^\dagger \rangle \qquad (3)$$

$$= -[1 + \zeta n_\zeta(\varepsilon_k)]\, \frac{e^{i\omega_n\beta} e^{-\varepsilon_k\beta} - 1}{i\omega_n - \varepsilon_k} \qquad (4)$$

Can we analytically continue the expression (63.4)? $\boxed{GF64}$

$$\mathcal{G}_k(z) \overset{?}{=} -[1+\xi n_\xi(\varepsilon_k)]\, e^{z\beta}\, \frac{e^{-\varepsilon_k\beta}-1}{z-\varepsilon_k} \tag{1}$$

NO! because (iii) is violated: $z\,\mathcal{G}_k(z) \xrightarrow{|z|\to\infty} e^{z\beta}$, which diverges for $z\to\infty$ (2)

Before continuation, set $e^{i\omega_n\beta}=\xi$ in (1):

$$\mathcal{G}_k(i\omega_n) = \underbrace{-[1+\xi n_\xi(\varepsilon_k)]}\,[\cancel{\xi\, e^{-\varepsilon_k\beta}}-1]\frac{1}{i\omega_n-\varepsilon_k} = \frac{1}{i\omega_n-\varepsilon_k} \tag{3}$$

$$-\langle c_k c_k^\dagger\rangle = -1-\frac{\xi}{e^{\beta\varepsilon_k}-\xi} = \frac{-e^{\beta\varepsilon_k}}{e^{\beta\varepsilon_k}-\xi} = \frac{-1}{1-\cancel{\xi e^{-\beta\varepsilon_k}}} = \xi\, n_\xi(-\varepsilon_k) \tag{4}$$

Now we can analytically continue: $\mathcal{G}_k(z) = \dfrac{1}{z-\varepsilon_k}$ (5)

which does satisfy (iii): $z\,\mathcal{G}_k(z) \xrightarrow{|z|\to\infty} 1$ (6)

In practice, (iii) means: replace $e^{i\beta\omega_n}=\xi$ before analytic continuation!

Matsubara sums $\boxed{GF65}$

One often needs to perform sums over all Matsubara frequencies, of the form

$$S = \frac{1}{\beta}\sum_n F(i\omega_n), \quad \text{where } F(z) \text{ has simple poles at } z\in\{\omega_a\} \text{ with residues } R(\omega_a) \tag{1}$$

Example: inverse Matsubara transformation:

$$\mathcal{G}_k(\tau) \overset{(23.2)}{=} \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}\,\mathcal{G}_k(i\omega_n) \overset{(64.3)}{=} \frac{1}{\beta}\sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n-\varepsilon_k} = \,? \tag{2}$$

General method: convert sum to contour integral (using function with poles at $i\omega_n$)

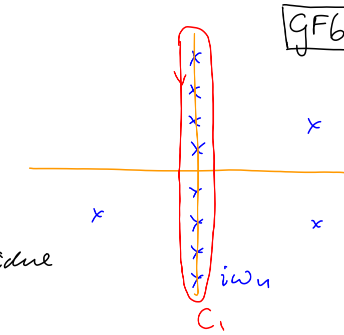
The function $n_\xi(\pm z) = \dfrac{1}{e^{\pm\beta z}-\xi}$ (3)

has poles at $z=i\omega_n$ (since $e^{\pm i\omega_n\beta}=\xi$), with residue $\dfrac{1}{\pm\xi\beta}$ (4)

since $$\lim_{\delta\to 0}\left[\delta\, n_\xi(\pm(i\omega_n+\delta))\right] = \lim_{\delta\to 0}\left[\frac{\delta}{e^{\pm i\omega_n\beta}e^{\pm\beta\delta}-\xi}\right] = \lim_{\to 0}\frac{\cancel{\delta}}{\xi(\cancel{1}\pm\beta\cancel{\delta})-\cancel{\xi}} = \frac{1}{\pm\xi\beta} \tag{5}$$

So: write $S \overset{(65.1)}{=} \frac{1}{\beta} \sum_n F(i\omega_n)$ (1)

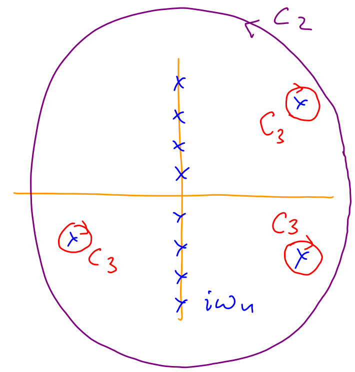
$$= \frac{1}{\beta}(\pm \xi \beta) \oint_{C_1} \frac{dz}{2\pi i}\, n_\xi(\pm z)\, F(z) \qquad (2)$$



The contour integration around C_1 in (2) will, by residue theorem, precisely reproduce the sum in (1).

$$\oint_{C_1} dz = \oint_{C_2} dz + \oint_{C_3} dz \qquad (3)$$

$\oint_{C_3}$ ← encircles (clockwise) all poles ω_a of $F(z)$



C_2 yields 0 if $\lim_{|z|\to\infty} \left| z\, n_\xi(\pm z)\, F(z) \right| = 0$ (4)

Requirement (4) usually fixes choice of sign, $\pm$

Residue theorem for C_3:
$$S \overset{(2)}{=} -(\pm \xi) \sum_a n_\xi(\pm \omega_a)\, R(\omega_a) \qquad (5)$$

Example 1: $\mathcal{G}_k(\tau) \overset{(65.2)}{=} \frac{1}{\beta} \sum_n \frac{e^{-i\omega_n \tau}}{i\omega_n - \varepsilon_k}$, (1)

here $F(z) = \frac{e^{-z\tau}}{z - \varepsilon_k}$ has simple pole at $\omega_a = \varepsilon_k$, with residue $R(\omega_a) = e^{-\varepsilon_k \tau}$ (2)

For $\left\{ \begin{matrix} -\beta < \tau < 0 \\ 0 < \tau < \beta \end{matrix} \right\}$ use $\left\{ \begin{matrix} \text{upper} \\ \text{lower} \end{matrix} \right\}$ sign in $n_\xi(\pm z)$ to satisfy condition (66.4): (3)

For $-\beta < \tau < 0$: $\lim_{|z|\to\infty} \left| \frac{z}{e^{+\beta z} - \xi} \frac{e^{+z|\tau|}}{z - \varepsilon_k} \right|$ $\xrightarrow{z\to\infty} e^{|\tau| z} e^{-\beta z} \to 0$; $\xrightarrow{z\to -\infty} e^{-|\tau||z|} \to 0$ (4)

$$\mathcal{G}_k(\tau) \overset{(66.5)}{=} -\xi\, n_\xi(\varepsilon_k)\, e^{-\varepsilon_k \tau} = -\xi \langle c_k^\dagger(0)\, c_k(\tau) \rangle \qquad (5)$$

For $0 < \tau < \beta$: $\lim_{|z|\to\infty} \left| \frac{z}{e^{-\beta z} - \xi} \frac{e^{-z\tau}}{z - \varepsilon_k} \right|$ $\xrightarrow{z\to\infty} e^{-z\tau} \to 0$; $\xrightarrow{z\to -\infty} e^{\tau|z|} e^{-\beta|z|} \to 0$ (6)

$$\mathcal{G}_k(\tau) \overset{(66.5)}{=} +\xi\, n_\xi(-\varepsilon_k)\, e^{-\varepsilon_k \tau} = -\langle c_k(\tau)\, c_k^\dagger(0) \rangle \qquad (7)$$

Remarkably, Matsubara summation in (1) correctly yields both branches, $\mathcal{G}(\tau \gtrless 0)$!

Example 2: when calculating density response, one needs:

$$S(i\omega_m) = \frac{1}{\beta}\sum_n \frac{1}{i\omega_n - \omega}\,\frac{1}{i\omega_n + i\omega_m - \omega'} \quad (1) \qquad \text{where } \omega_m = \text{bosonic} \quad (2)$$

$$F(z) = \frac{1}{z-\omega}\,\frac{1}{z+i\omega_m-\omega'}, \quad \text{poles at } \begin{cases} z = \omega, & R = \dfrac{1}{\omega + i\omega_m - \omega'} \\ z = \omega' - i\omega_m & R = \dfrac{1}{\omega' - i\omega_m - \omega} \end{cases} \quad (3)$$

$$\lim_{z\to\infty} |z\,F(z)\,n_\xi(\pm z)| = 0 \quad \text{for both signs, since } F(z) \to \frac{1}{|z|^2}. \quad (4)$$

So, we can use either sign. Lets keep track of both possibilities.

$$S(i\omega_m) = \mp\xi\left[\frac{n_\xi(\pm\omega)}{\omega + i\omega_m - \omega'} + \frac{\overbrace{n_\xi(\pm(\omega' - i\omega_m))}^{= n_\xi(\pm\omega') \text{ since } e^{i\omega_m\beta} \overset{(2)}{=} 1}}{\omega' - i\omega_m - \omega}\right] \quad (5)$$

$$= \mp\xi\,\frac{n_\xi(\pm\omega) - n_\xi(\pm\omega')}{i\omega_m - \omega' + \omega} \quad (6)$$

upper/lower signs give same answer, since $n_\xi(-\omega) \overset{(64.4)}{=} -\xi - n_\xi(\omega)$ (7)