Perturbation Theory for a Single-Particle Potential (Richayzen, §3.1) $\boxed{\text{PT 1}}$

Consider single-particle Hamiltonian: $h(x,\tau) = h_0(x) + U(x,\tau)$ (1)

$$h_0(x) = \underbrace{-\frac{\hbar^2\vec{\nabla}^2}{2m}}_{\text{kinetic energy}} \underbrace{-\mu}_{\text{chemical potential}} + \underbrace{\sum_i V_{imp}(\vec{r}-\vec{R}_i)}_{\text{impurity potential}} + \underbrace{h_\sigma(\vec{r})}_{\text{Zeeman term, present for fermions, depends on spin } \sigma} \quad (2)$$

$\vec{R}_i$: impurity positions

Perturbation: $U(\vec{r},\tau)$ = external (scalar potential) due to an applied field (e.g. electric field for fermions, or optical trap for bosonic atoms) (3)

Notation: $x \equiv$ shorthand for $(\vec{r},\sigma)$, $\sigma \in \{\uparrow,\downarrow\}$ = spin (4)

$$\int dx \equiv \sum_\sigma \int d\vec{r} \quad (5)$$

$$\delta(x_1 - x_2) = \delta_{\sigma_1\sigma_2}\,\delta(\vec{r}_1 - \vec{r}_2) \quad (6)$$

Let $\varphi_\lambda(\vec{r})$ be single-particle eigenstates of h_0: $\boxed{\text{PT 2}}$

$$h_0(x)\,\varphi_\lambda(\vec{r}) = \varepsilon_\lambda\,\varphi_\lambda(\vec{r}) \quad (1)$$

orthogonal: $\int dx\, \varphi^*_\lambda(x)\,\varphi_{\lambda'}(x) = \delta_{\lambda\lambda'}$ (2)

complete: $\sum_\lambda \varphi_\lambda(x)\,\varphi^*_\lambda(x') = \delta(x-x')$ (3)

Many-body formulation (in Schrödinger picture):

$$\hat{H}(\tau) = \hat{H}_0 + \hat{H}'(\tau) \quad (4)$$

$$\hat{H}_0 = \int dx\, \hat{\psi}^\dagger(x)\, h_0(x)\, \hat{\psi}(x) \quad (5)$$

$$\hat{H}'(\tau) = \int dx\, \hat{\psi}^\dagger(x)\, U(x,\tau)\, \hat{\psi}(x) \quad (6)$$

PT 3

with field operators: $\underline{\hat{\psi}(x)} \equiv \sum_\lambda \underline{\varphi_\lambda(x)\, c_\lambda}$ (1)

where $[c_\lambda, c^\dagger_{\lambda'}]_\zeta = \delta_{\lambda\lambda'}$, (2)

hence $\underline{[\hat{\psi}(x), \hat{\psi}^\dagger(x')]_\zeta} \overset{(1)}{=} \sum_{\lambda\lambda'} \overbrace{[c_\lambda, c^\dagger_{\lambda'}]_\zeta}^{(8)\ \delta_{\lambda\lambda'}} \varphi_\lambda(x)\, \varphi^*_{\lambda'}(x')$ (3)

$\overset{(3)}{=} \underline{\delta(x-x')}$ "equal-time commutator" (4)

then $\hat{H}_0 = \sum_{\lambda\lambda'} \int dx\; \varphi^*_\lambda(x)\, \underbrace{h_0(x)\, \varphi_{\lambda'}(x)}_{(2.1)\ \varepsilon_{\lambda'}\varphi_{\lambda'}(x)}\; c^\dagger_\lambda c_{\lambda'}$ (5)

$\overset{(2.2)}{=} \sum_\lambda \varepsilon_\lambda c^\dagger_\lambda c_\lambda$ (familiar form used so far) (6)

PT 4

Goal: find single-particle GF in presence of $\hat{H}'(\tau)$.

Go to Heisenberg picture: $-\partial_\tau \hat{A}(\tau) = [\hat{A}(\tau), \hat{H}(\tau)]$ (1)

$\hat{\psi}(1) \equiv \hat{\psi}(\underbrace{x,\tau}_{\equiv(1)}) = \sum_\lambda \varphi_\lambda(x)\, c_\lambda(\tau)$ (2)

eq. of motion for field:

$-\partial_{\tau_1} \hat{\psi}(1) \overset{(1)}{=} [\hat{\psi}(1), \hat{H}(\tau)]$ (3)

$\overset{(2,5,6)}{=} \int dx' \; [\underbrace{\hat{\psi}(x,\tau), \hat{\psi}^\dagger(x',\tau)}_{(3.4)\ \delta(x-x')} (h_0(x') + U(x',\tau))\, \hat{\psi}(x',\tau)]$ (4)

$= [h_0(x) + U(x,\tau)]\, \hat{\psi}(x,\tau)$ (5)

shorthand: $= [h_0(1) + U(1)]\, \hat{\psi}(1)$ (6)

Thermal GF: $\mathcal{G}(1,2) \overset{(GF20.1)}{\equiv} -\langle T_\tau\, \hat\psi(1)\, \hat\psi^\dagger(2)\rangle$ (1) $\boxed{\text{PT 5}}$

$$-\partial_{\tau_1}\mathcal{G}(1,2) \overset{(\text{compare } GF20.3)}{=} \delta(\tau_1-\tau_2)\,\langle\, \underbrace{(\hat\psi(1),\hat\psi^\dagger(2)) - \xi(\psi^\dagger(\ ),\psi(\ ))}_{[\hat\psi(1),\hat\psi^\dagger(2)]_\xi \overset{(3.4)}{=} \delta(x_1-x_2)}\,\rangle \qquad (2)$$

since $\tau_1=\tau_2$

$$+\,[\overset{(4.6)}{h_0(1)} + U(1)]\,\mathcal{G}(1,2)$$

$$\Longrightarrow \quad [-\partial_{\tau_1} - h_0(1) - U(1)]\,\mathcal{G}(1,2) = \delta(\tau_1-\tau_2)\,\delta(x_1-x_2) \equiv \delta(1,2) \qquad (3)$$

We seek expansion of $\mathcal{G}$ in powers of U: $\mathcal{G} = \sum_{n=0}^{\infty}\mathcal{G}_0$ (4)

$\mathcal{G}_0$ = "unperturbed $\mathcal{G}$" satisfies: $[-\partial_{\tau_1} - h_0(1)]\,\mathcal{G}_0(1,2) \overset{U=0 \text{ in } (4)}{=} \delta(1,2)$ (5)

(1) and (3) are to be solved for $0 \le \tau_1, \tau_2 \le \beta$,

with boundary condition $\mathcal{G}(x_1, 0; 2) \overset{(GF22.4)}{=} \xi\,\mathcal{G}(x_1,\beta;2)$ (6)

Using (5.5), we can construct general solution of (5.3): $\boxed{\text{PT 6}}$

$$\mathcal{G}(1,2) = \mathcal{G}_0(1,2) + \int_0^\beta d3\; \mathcal{G}_0(1,3)\, U(3)\, \mathcal{G}(3,2) \qquad (1)$$

with $\int_0^\beta d3 \equiv \int_0^\beta d\tau_3 \int dx_3$ (2)

since $[-\partial_{\tau_1} - h_0(1)]$ (1) yields:

$$[-\partial_{\tau_1} - h_0(1)]\,\mathcal{G}(1,2) \overset{(1)}{=} \overset{(5.5)}{\delta(1,2)} + \underbrace{\int_0^\beta d3\; \overset{(5.5)}{\delta(1,3)}\, U(3)\, \mathcal{G}(3,2)}_{=\,U(1)\,\mathcal{G}(1,2)} \;\Rightarrow\; \text{yields (5.3)} \checkmark \qquad (3)$$

boundary condition (5.6) is satisfied by $\mathcal{G}_0$, hence also by $\mathcal{G}$ ✓.

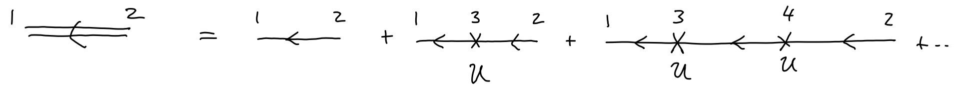
Iterating (1):

$$\mathcal{G}(1,2) = \mathcal{G}_0(1,2) + \int d3\; \mathcal{G}_0(1,3)\,U(1)\,\mathcal{G}_0(3,2) + \int d3\int d4\; \mathcal{G}_0(1,3)\,U(1)\,\mathcal{G}_0(3,4)\,U(4)\,\mathcal{G}_0(4,2) + \dots \qquad (4)$$

$\mathcal{G} = \sum_n \mathcal{G}_n$,

$$\text{with } \mathcal{G}_n(1,2) = \int d3\int d4\; \mathcal{G}_0(1,3)\,U(1)\,\mathcal{G}_0(3,4)\,U(4) \times \dots\; U(2+n)\,\mathcal{G}(n+2,2) \qquad (5)$$

Graphical notation for (6.4)



(1)

$$\mathcal{G}(1,2) = -\theta(\tau_1-\tau_2)\langle\psi(1)\,\psi^+(2)\rangle - \zeta\,\theta(\tau_2-\tau_1)\langle\psi^+(2)\,\psi(1)\rangle \qquad (2)$$

(τ increases from right to left)

particle destroyed particle created

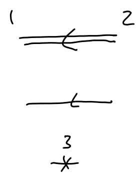
particle created particle destroyed

hole destroyed hole created

$\tau_1 > \tau_2$: particle propagation from $2 \to 1$ (forward in time) (3)

$\tau_2 > \tau_1$: $\begin{cases} \text{particle propagation from } 2 \to 1 & \text{(backward in time)} \\ \text{hole propagation from } 1 \to 2 & \text{(forward in time)} \end{cases}$ } equivalent views (4)

Shorthand: $\mathcal{G}(1,2)$:

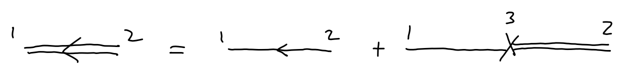


$\mathcal{G}_0(1,2)$:

$\mathcal{U}(3)$:

"Feynman rule": integrate over all internal vertices with $\int d3 = \int d\tau_3 \int dx_3$ (5)

(1) implies: "Dyson eq."



which gives (6.3) (6)

Interacting particles

$$\hat{H} = \hat{H}_0 + \hat{H}' \qquad (1)$$

$$\hat{H}_0 = \int dx\; \hat{\psi}^+(x)\; h_0(x)\; \hat{\psi}(x) \qquad (2)$$

$$h_0(x_1) = -\frac{\nabla^2}{2m} - \mu + \mathcal{U}(x_1) \qquad (3)$$

external potential, and impurities

$$\hat{H}' = \frac{1}{2}\int dx \int dx'\; \psi^+(x)\,\psi^+(x')\; \underbrace{V(x,x')}_{\text{typically } = V(\vec{r}-\vec{r}\,'),\text{ and time-independent}}\; \psi(x')\,\psi(x) \qquad (4)$$

$\mathcal{G}_0 \equiv$ GF for $\hat{H}_0$, including $\mathcal{U}$ (so $\mathcal{G}_0 = \mathcal{G}$ from above)

We want to develop perturbation theory of the type