Wick's theorem in graphical practice

PT30

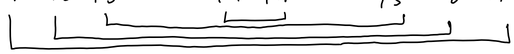
$$\mathcal{G}_0^{(n)}(1,\dots,n;n',\dots,1') = \sum_{\text{all permutations}} \zeta^P \, \mathcal{G}_0^{(1)}(1,P(1)) \dots \mathcal{G}_0^{(1)}(n,P(n))$$

$^{(1)}$ ← will be dropped henceforth

graphically:

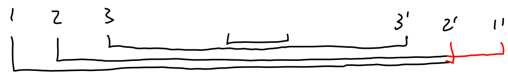
just the right sign for n one-point functions, $(\mathcal{G}_0)^n$

$$(-)^n \langle \psi_1 \psi_2 \psi_3 \dots \psi_n \psi^\dagger_{n'} \dots \psi^\dagger_{3'} \psi^\dagger_{2'} \psi^\dagger_{1'} \rangle$$



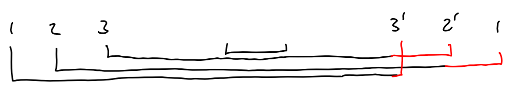
$$\Rightarrow (-)\langle \psi_1 \psi^\dagger_{1'}\rangle (-)\langle \psi_2 \psi^\dagger_{2'}\rangle (-)\langle \psi_3 \psi^\dagger_{3'}\rangle$$
$$= \mathcal{G}_0(1,1')\,\mathcal{G}_0(2,2')\,\mathcal{G}_0(3,3')$$

one switch: ζ



$$+ \zeta (-)\langle \psi_1 \psi^\dagger_{2'}\rangle (-)\langle \psi_2 \psi^\dagger_{1'}\rangle \dots$$
$$= \zeta \; \mathcal{G}_0(1,2')\,\mathcal{G}_0(2,1')\,\mathcal{G}(3,3') \dots$$

two switches: ζ^2



$$+ \zeta^2 (-)\langle \psi_1 \psi^\dagger_{1'}\rangle (-)\langle \psi_2 \psi^\dagger_{2'}\rangle \dots$$
$$= \zeta^2 \; \mathcal{G}_0(1,3')\,\mathcal{G}_0(2,1')\,\mathcal{G}_0(3,2')$$

$+ \dots$

Diagrammatic perturbation theory

(Rickayzen, § 3.2)

PT31

Goal: diagrammatic rules for evaluating:

$$\mathcal{G}^{(1)}(1,2) \overset{(13.3)}{=} - \frac{\langle T_\tau \, \hat{U}_I(\beta)\, \psi_I(1)\, \psi_I^\dagger(2)\rangle_0}{\langle T_\tau \, \hat{U}_I(\beta)\rangle_0} \tag{1}$$

$$\hat{H}_0 \overset{(8.2)}{=} \int dx \, \psi^\dagger(x)\, h_0\, \psi(x) \tag{2}$$

$$\hat{H}' \overset{(8.4)}{=} \frac{1}{2}\int dx \int dx' \; \psi^\dagger(x)\,\psi^\dagger(x')\; v(x,x')\; \psi(x')\,\psi(x) \tag{3}$$

$$\int_0^\beta d\tau_1' \; H_I'(\tau_1') = \frac{1}{2}\int d1'' \int d1' \; \psi^\dagger(1''_+)\, \psi^\dagger(1'_+)\, v(1'',1')\, \psi(1')\, \psi(1'') \tag{4}$$

where $\int d1' \equiv \int_0^\beta d\tau_1' \int dx_1' = \int d\bar{r}_1 \sum_{\sigma_1}$, $\quad \psi^\dagger(1'_+) \equiv \psi_I^\dagger(\tau'_{1+\alpha}, x_1')$ with $\tau'_{1+\alpha} = \tau_1'^+$, $\quad v(1'',1') \equiv v(x_1'',x_1')\,\delta(\tau_1''-\tau_1')$, $\quad \psi(1') \equiv \psi_I(\tau_1',x_1')$

$_I$ ← will be dropped henceforth

Note: in Hamiltonian, we represent $\psi^\dagger(\tau_1)\,\psi(\tau_1)$ by $\psi^\dagger(\tau_{1+\alpha})\,\psi(\tau_1)$ (5)

(α: positive infinitesimal; $\tau_{1+\alpha} \equiv \tau_1^+$)

so that $T_\tau\left(\psi^\dagger(\tau_1^+)\,\psi(\tau_1)\right)$ automatically yields the correct order.

Rule of thumb: "creation operators are at slightly larger times than annihilation op."

Partition function:

PT32

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] \overset{(11.1)}{=} \mathrm{Tr}\left[e^{-\beta H_0}\, \mathcal{U}_I(\beta)\right] = \langle \mathcal{U}_I(\beta)\rangle_0 \qquad (1)$$

$$\overset{(11.2)}{=} \langle T_\tau\, e^{-\int_0^\beta d\tau_1\, H_I'(\tau_1)}\rangle_0 \qquad (2)$$

$$\overset{(13.4)}{=} \sum_{n=0}^{\infty} \frac{1}{n!}(-)^n \int_0^\beta d\tau_1 \dots \int_0^\beta d\tau_n \,\langle T_\tau\, H_I'(\tau_1) \dots H_I'(\tau_n)\rangle_0 \qquad (3)$$

$$\overset{(31.4)}{=} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\tfrac{1}{2}\right)^n \int d1'' \int d1' \dots \int dn'' \int dn' \; v(1'',1') \dots v(n'',n')$$
$$\times \langle T_\tau\, \psi^\dagger(1''_+)\, \psi^\dagger(1'_+)\, \psi(1')\, \psi(1'') \dots \psi^\dagger(n''_+)\, \psi^\dagger(n'_+)\, \psi(n')\, \psi(n'')\rangle_0 \qquad (4)$$

rearrange into two useful alternative orders (both get sign of (+1), since $(\psi\psi)$ or $(\psi^\dagger\psi^\dagger)$ are moved past each other pairwise)

$$= (-)^{2n} \langle T_\tau\, \psi^\dagger(1''_+)\, \psi(1'')\, \psi^\dagger(1'_+)\, \psi(1') \dots \psi^\dagger(n''_+)\, \psi(n'')\, \psi^\dagger(n'_+)\, \psi(n')\rangle \qquad (5)$$

insert factor 1

or
$$= (-)^{2n} \langle T_\tau\, \psi(1')\, \psi(1'') \dots \psi(n')\, \psi(n'')\, \psi^\dagger(n''_+)\, \psi^\dagger(n'_+) \dots \psi^\dagger(1''_+)\, \psi(1'_+)\rangle_0 \qquad (6)$$

Identify correlator as $\mathcal{G}_0^{2n}$:

PT33

$$Z \overset{(32.6)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \int d1'' \int d1' \dots \int dn'' \int dn' \left(-\tfrac{1}{2}\right)^n v(1'',1') \dots v(n'',n')$$
$$\times\, \mathcal{G}_0^{(2n)}(1',1'',\dots,n',n'';\, 1'_+,1''_+,\dots,n'_+,n''_+) \qquad (1)$$

$$\equiv \sum_{n=0}^{\infty} Z_{[n]} \qquad \text{(expansion in powers of } v\text{)} \qquad (2)$$

Consider lowest terms explicitly:

$$Z_{[0]} = 1 \qquad (3)$$

$$Z_{[1]} \overset{(32.5)}{=} \int d1'' \int d1' \left(-\tfrac{1}{2}\right) v(1'',1')\, (-)^2 \langle T_\tau\, \psi^\dagger(1''_+)\, \psi(1'')\, \psi^\dagger(1'_+)\, \psi(1')\rangle \qquad (4)$$

a, b, c, d

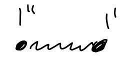
Wick

$$= \int d1'' \int d1' \left(-\tfrac{1}{2}\right) v(1'',1') \left[\xi^2\, \mathcal{G}_0(1'',1''_+)\, \mathcal{G}_0(1',1'_+) + \xi\, \mathcal{G}_0(1'',1'_+)\, \mathcal{G}_0(1',1''_+)\right] \qquad (5)$$

Diagrammatic shorthand:

(6)

Some Feynman rules: [PT34]

(1) Factor $-v(1'',1')$ for the interaction line  (1)

(2) $\mathcal{G}_0(1,2)$ for the particle propagator  (2)

(3) Factor ξ for each closed loop, because such a loop (3)

(say with m interaction lines) requires odd # of switches relative to original order in (32.5):

odd number of permutations : ξ

$$(\psi^\dagger_{1''}\psi_{1''})(\psi^\dagger_{1'}\psi_{1'}) \;\dots\; (\psi^\dagger_{m''}\psi_{m''})(\psi^\dagger_{m'}\psi_{m'}) \tag{4}$$

$$= \xi\,(-)^{2m}\; \psi_{1''}(\psi^\dagger_{1'}\psi_{1'})(\psi^\dagger_{2''}\psi_{2''})\dots(\psi^\dagger_{m''}\psi_{m''})(\psi^\dagger_{m'}\psi_{m'})\,\psi^\dagger_{1''} \tag{5}$$

factor 1 — 2m contractions, unentangled

$$= \xi\, \mathcal{G}_0(1'',1'_+)\,\mathcal{G}_0(1',2''_+)\dots\mathcal{G}(m'',m'_+)\,\mathcal{G}(m',1''_+) \tag{6}$$

(4) Extra overall sign: none, since each $H_I \sim \psi^\dagger\psi^\dagger\psi\psi \sim \psi\psi\psi^\dagger\psi^\dagger \sim (-\mathcal{G}_0)^2$ yields two factors of $\mathcal{G}_0$, with no need for extra signs. (7)

Analogously consider numerator of (31.1): [PT35]

$$-\langle T_\tau\, \hat{U}_I(\beta)\, \psi_I(1)\, \psi_I^\dagger(2)\rangle_0 \tag{1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d1'' \int d1' \dots \int dn'' \int dn' \left(-\tfrac{1}{2}\right)^n v(1'',1') \dots v(n'',n') \tag{2}$$

as in (32.5):

$$\times (-)^{2n+1} \langle T_\tau\, \psi(1)\, \psi^\dagger(1''_+)\psi(1'')\,\psi^\dagger(1'_+)\psi(1') \dots \psi^\dagger(n''_+)\psi(n'')\,\psi^\dagger(n'_+)\psi(n')\,\psi^\dagger(2)\rangle_0$$

or (32.6):

$$= (-)^{2n+1} \langle T_\tau\, \psi(1)\,\psi(1')\,\psi(1'')\dots\psi(n')\,\psi(n'')\,\psi^\dagger(n''_+)\,\psi^\dagger(n'_+)\dots\psi^\dagger(1''_+)\psi(1'_+)_0\,\psi^\dagger(2)\rangle_0 \tag{3}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d1'' \int d1' \dots \int dn'' \int dn' \left(-\tfrac{1}{2}\right)^n v(1'',1') \dots v(n'',n')$$

$$\times\, \mathcal{G}_0^{(2n+1)}(1, 1', 1'', \dots, n', n''; 2, 1'_+, 1''_+, \dots, n'_+, n''_+) \tag{4}$$

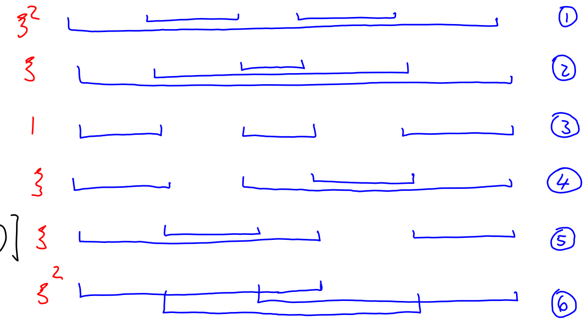
$$= \sum_{n=0}^{\infty} \mathcal{G}_{[n]}(1,2) \tag{5}$$

← indicates: n powers of v

$$\mathcal{G}_{[0]}(1,2) = \mathcal{G}_0^{(1)}(1,2) \tag{6}$$

PT36

$$\mathcal{G}_{[1]}(1,2) = \int d1'' \int d1' \left(-\tfrac{1}{2} v(1'',1')\right) (-)^3 \langle T_\tau\, \psi(1)\; \psi^\dagger(1''_+)\psi(1'')\; \psi^\dagger(1'_+)\psi(1')\; \psi^\dagger(2) \rangle \quad (1)$$



$$= \int d1'' \int d1' \left(-\tfrac{1}{2} v(1'',1')\right) \left[① + ② + ③ + ④ + ⑤ + ⑥ \right] \quad (2)$$

①+②: $\quad \mathcal{G}_0(1,2') \left[\mathcal{G}_0(1'',1''_+)\, \mathcal{G}_0(1',1'_+) + \xi\, \mathcal{G}_0(1'',1'_+)\, \mathcal{G}_0(1',1''_+) \right]$ (3)

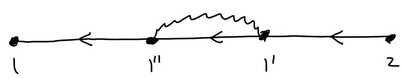
as in (33.6):

(4)

These are "disconnected" diagrams, since they factorize into two factors, one of which is independent of external vertices (1 and 2).

PT37

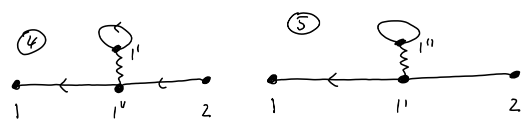
③: $\quad \mathcal{G}_0(1,1''_+)\, \mathcal{G}_0(1'',1'_+)\, \mathcal{G}_0(1',2)$



(1)

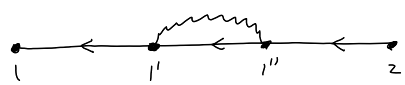
④: $\quad \xi\, \mathcal{G}_0(1,1''_+)\, \mathcal{G}_0(1'',2)\, \mathcal{G}_0(1',1'_+)$

⑤: $\quad \xi\, \mathcal{G}_0(1,1'_+)\, \mathcal{G}_0(1',2)\, \mathcal{G}_0(1'',1''_+)$



(2)

⑥: $\quad \mathcal{G}_0(1,1'_+)\, \mathcal{G}_0(1',1''_+)\, \mathcal{G}_0(1'',2)$



(3)

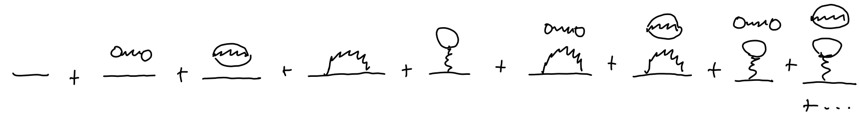
- Factor ξ for every loop. (confirming 34.3) (4)
- by relabelling integration variables $1' \leftrightarrow 1''$, we see: ③ = ⑥, ④ = ⑤ (5)
- In general: Faktor $\frac{1}{2}$ in H' is usually compensated by factor 2 from [exception: disconnected diag, (33.6)(36.4)] $\int d1'' \int d1' = \int d1' \int d1''$. Graphical argument: external line can always be attached in two ways: (6)

(7)

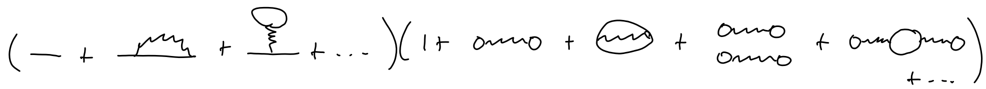
- Extra overall sign: none, since $-\langle \psi_1 \psi_2^\dagger \rangle \sim (-)^{2n+1} \langle \psi_1 (\psi^\dagger\psi\, \psi^\dagger\psi)^n \psi_2^\dagger \rangle \sim (\mathcal{G}_0)^{2n+1}$

- "Connected diagrams": <u>cannot</u> be separated into two parts without cutting one line (1)
- "Disconnected diagrams": <u>can</u> be separated " " " " " " " (2)

$\Rightarrow$ factorizes completely into two factors.

$-\langle T_\tau\, U\, \psi_1 \psi_2^\dagger \rangle =$ 

$$= \sum_n \frac{1}{2^n}\frac{1}{n!}\left(\Sigma \text{ all diagrams of order } n\right)$$

factorize: = 

$$= \left(\Sigma \text{ all connected diagrams}\right)\left[\sum_p \frac{1}{2^p\, p!}\left(\Sigma \text{ all disconnected diagrams of order } p\right)\right]$$

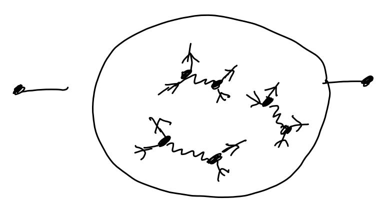
$\equiv C$ (compare p. 33) $\equiv D = Z = \langle U_I \rangle_0$

$$\Rightarrow \quad \mathcal{G} = -\frac{\langle T_\tau U_I \psi_1 \psi_2^\dagger\rangle_0}{\langle U_I\rangle_0} = \frac{C\cdot D}{Z} = C = \left(\Sigma \text{ all connected diagrams}\right)$$

with prefactor 1 for all topologically distinguishable diagrams.

Combinatorical factors:

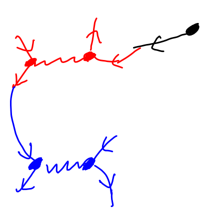
A. Diagram of order $(H')^n$, (i.e. $2n$ vertices) fully connected:



Combinatorical factor 1, since:

$$\underbrace{\left(\frac{1}{2}\right)^n \frac{1}{n!}} \times \left(2^n\, n!\right)$$ ← number of ways, to construct the "backbone" of the diagram:

From $T_\tau e^{-i\int d\tau''\int d\tau' \frac{1}{2} v \ldots}$



- Connect to 1st vertex $2n$ possibilities
- Connect to 2nd vertex $2n-1$ "

⋮

- Connect to n-th vertex $2\cdot 1$ "