Wick's theorem (Fetter & Walecka, p. 237–241)

PT16

Consider quadratic Hamiltonian: $\hat{H}_0 = \sum_\lambda \varepsilon_\lambda c^\dagger_\lambda c_\lambda$ (1)

Shorthand: $\hat{\alpha}_j \equiv c_{\lambda_j}$ or $c^\dagger_{\lambda_j}$ (2)

Consider thermal expectation value w.r.t. H_0, (denoted by $\langle \dots \rangle_0 = \frac{\mathrm{Tr}\, e^{-\beta H_0} \dots}{Z_0}$) of n creation and n annihilation operators: (3)

Wick: $$\langle \hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_{2n} \rangle_0 = \sum \Big[\underbrace{\alpha_1 \alpha_2}\, \underbrace{\alpha_3 \alpha_4} \dots \underbrace{\alpha_{2n-1} \alpha_{2n}}$$
$$+ \alpha_1 \alpha_2 \alpha_3 \alpha_4 \dots \underbrace{\alpha_{2n-1} \alpha_{2n}}$$
$$+ \dots \Big] \quad (4)$$

all possible pairwise contractions

where "contractions" are defined by $\underbrace{\alpha_i \alpha_j} \equiv \langle \alpha_i \alpha_j \rangle_0$ (5)

and interchanging contracted operators produces a sign: $\alpha_i \alpha_j \alpha_k \alpha_l = \zeta\, \underbrace{\alpha_i \alpha_k}\, \underbrace{\alpha_j \alpha_l}$ (6)

Proof:

PT17

$$[\hat{\alpha}_i, \hat{\alpha}_j]_\zeta = \hat{\alpha}_i \hat{\alpha}_j - \zeta \hat{\alpha}_j \hat{\alpha}_i = \begin{cases} \delta_{ij} & \text{for } [c_i, c_j^\dagger]_\zeta \\ (-\zeta)\delta_{ij} & \text{for } [c_i^\dagger, c_j]_\zeta \\ 0 & \text{for } [c,c] \text{ or } [c^\dagger, c^\dagger] \end{cases} \quad (1)$$

$$\hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_{2n} \quad (2)$$

$$= [\hat{\alpha}_1, \hat{\alpha}_2]_\zeta \hat{\alpha}_3 \dots \alpha_{2n} + \zeta\, \underbrace{\hat{\alpha}_2 \hat{\alpha}_1 \hat{\alpha}_3 \dots \hat{\alpha}_{2n}}$$

$$= \alpha_2 [\hat{\alpha}_1, \hat{\alpha}_3]_\zeta \hat{\alpha}_4 \dots \hat{\alpha}_n + \zeta\, \underbrace{\hat{\alpha}_2 \hat{\alpha}_3 \hat{\alpha}_1 \hat{\alpha}_4 \dots \hat{\alpha}_{2n}} \quad (3)$$

etc. after $2n-1$ swaps we get:

$$= \sum_{j=2}^{2n} \zeta^{j-2} \hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_{j-1} [\hat{\alpha}_1, \hat{\alpha}_j]_\zeta \hat{\alpha}_{j+1} \dots \hat{\alpha}_{2n} \quad (4)$$

$$+ \underbrace{\zeta^{2n-1}}_{=\zeta} \hat{\alpha}_2 \hat{\alpha}_3 \dots \hat{\alpha}_{2n} \hat{\alpha}_1$$

PT18

Now
$$c_\lambda e^{-\beta H_0} = e^{-\beta H_0} e^{-\beta\varepsilon_\lambda} c_\lambda \quad (1)$$
$$c_\lambda^\dagger e^{-\beta H_0} = e^{-\beta H_0} e^{\beta\varepsilon_\lambda} c_\lambda^\dagger \quad (2)$$

compact notation:
$$\hat\alpha_1 e^{-\beta H_0} = e^{-\beta H_0} e^{(\beta\eta_1\lambda_1)} \hat\alpha_1 \quad \text{with } \eta_1 = \begin{cases} +1 & \text{for } \hat\alpha = c^\dagger \\ -1 & \text{for } \hat\alpha = c \end{cases} \quad (3)$$

So
$$\langle \hat\alpha_2\hat\alpha_3 \dots \hat\alpha_{2n}\hat\alpha_1\rangle = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H_0}\hat\alpha_2 \dots \hat\alpha_{2n}\hat\alpha_1\right) \quad (4)$$
$$= e^{\beta\eta_1\varepsilon_{\lambda_1}} \frac{1}{Z}\mathrm{Tr}\, e^{-\beta H_0}\hat\alpha_1\hat\alpha_2\dots\hat\alpha_{2n} = e^{\eta_1\beta\varepsilon_{\lambda_1}}\langle\hat\alpha_1\hat\alpha_2\dots\hat\alpha_{2n}\rangle \quad (5)$$

So: $\langle(17.4)\rangle$ yields:
$$\langle\hat\alpha_1\hat\alpha_2\dots\hat\alpha_n\rangle_0 (1 - \xi e^{\beta\eta_1\lambda_1}) = \quad (6)$$
$$= \sum_{j=2}^{2n} \xi^{j-2} \langle\hat\alpha_2\hat\alpha_3\dots\hat\alpha_{j-1}[\hat\alpha_1,\hat\alpha_j]_\xi\,\hat\alpha_{j+1}\dots\hat\alpha_{2n}\rangle_0$$

PT19

$$\langle\hat\alpha_1\dots\hat\alpha_n\rangle_0 = \sum_{j=2}^{2n}\xi^{j-2}\langle\hat\alpha_2\hat\alpha_3\dots\hat\alpha_{j-1}\frac{[\hat\alpha_1,\hat\alpha_j]_\xi}{1-\xi e^{\beta\eta_1\lambda_1}}\hat\alpha_{j+1}\dots\hat\alpha_{2n}\rangle_0 \quad (1)$$

Now
$$\langle c_\lambda^\dagger c_\lambda\rangle_0 = \frac{1}{e^{\beta\varepsilon_\lambda}-\xi} = \frac{-\xi}{1-\xi e^{\beta\varepsilon_\lambda}} \quad (2)$$
$$\langle c_\lambda c_\lambda^\dagger\rangle_0 = 1 + \xi\langle c_\lambda^\dagger c_\lambda\rangle_0 = \frac{e^{\beta\varepsilon_\lambda}-\xi+\xi}{e^{\beta\varepsilon_\lambda}-\xi} = \frac{1}{1-\xi e^{-\beta\varepsilon_\lambda}} \quad (3)$$

and, $[\alpha_1,\alpha_j]$ is nonzero only for $1=j$, giving:

- $$\frac{[c_{\lambda_1}, c^\dagger_{\lambda_j}]_\xi}{1-\xi e^{-\beta\varepsilon_{\lambda_1}}} = \delta_{ij}\langle c_{\lambda_1} c^\dagger_{\lambda_1}\rangle_0 = \langle c_{\lambda_1} c^\dagger_{\lambda_j}\rangle_0 \equiv \underbrace{c_{\lambda_1}\, c^\dagger_{\lambda_j}} \quad (4)$$
"contractions"
- $$\frac{[c^\dagger_{\lambda_1}, c_{\lambda_j}]_\xi}{1-\xi e^{\beta\varepsilon_{\lambda_1}}} = \delta_{ij}(-\xi)\cdot(-\xi)\langle c^\dagger_{\lambda_1} c_{\lambda_1}\rangle_0 = \langle c^\dagger_{\lambda_1} c_{\lambda_j}\rangle_0 \equiv \underbrace{c^\dagger_{\lambda_1}, c_{\lambda_j}} \quad (5)$$

Similarly, define
$$\underbrace{c_{\lambda_1} c_{\lambda_2}} \equiv \langle c_{\lambda_1} c_{\lambda_2}\rangle_0 = 0 \quad (6)$$
$$\underbrace{c^\dagger_{\lambda_1} c^\dagger_{\lambda_2}} \equiv \langle c^\dagger_{\lambda_1} c^\dagger_{\lambda_2}\rangle_0 = 0 \quad (7)$$

PT20

$$\langle \hat\alpha_1 \dots \hat\alpha_n \rangle_0 \overset{(19.1)}{=} \sum_{j=2}^{2n} \zeta^{j-2} \langle \hat\alpha_2 \hat\alpha_3 \dots \hat\alpha_{j-1} \; \overset{(19.4,5)}{\underbrace{\alpha_1 \, \alpha_j}} \; \hat\alpha_{j+1} \dots \hat\alpha_{2n} \rangle_0 \qquad (1)$$

this is just a number, see (19.7)

convention:

$$\hat\alpha_i \underbrace{\alpha_j \alpha_k} \hat\alpha_l \equiv \zeta \, \alpha_j \hat\alpha_i \alpha_k \hat\alpha_l \equiv \zeta \, \hat\alpha_i \alpha_j \hat\alpha_l \alpha_k = \alpha_j \hat\alpha_i \hat\alpha_l \alpha_k \qquad (2)$$

i.e.: when moving either endpoint of a contraction past an operator, this gives a sign ζ. Then

$$\langle \hat\alpha_1 \dots \hat\alpha_n \rangle_0 = \sum_{j=2}^{2n} \langle \alpha_1 \hat\alpha_2 \hat\alpha_3 \dots \alpha_j \dots \hat\alpha_{2n} \rangle_0$$

or

$$= \sum_{j=2}^{2n} \underbrace{\alpha_1 \alpha_j} \; \zeta^{j-2} \langle \hat\alpha_2 \hat\alpha_3 \dots \hat\alpha_{j-1} \; \hat\alpha_{j+1} \dots \hat\alpha_{2n} \rangle_0$$

repeat the argument for remaining expectation value

PT21

$$\boxed{\langle \hat\alpha_1 \hat\alpha_2 \dots \hat\alpha_{2n} \rangle_0 = \sum \Big[\underbrace{\hat\alpha_1 \hat\alpha_2} \, \underbrace{\hat\alpha_3 \hat\alpha_4} \dots \underbrace{\hat\alpha_{2n-1} \hat\alpha_{2n}} + \hat\alpha_1 \hat\alpha_2 \hat\alpha_3 \hat\alpha_4 \dots \underbrace{\hat\alpha_{2n-1} \hat\alpha_{2n}} + \dots \Big] \quad \square}$$

all possible pairwise contractions

Wick's theorem for thermal expectation values. (1)

Careful: keep track of signs!

$$\alpha_1 \alpha_2 \alpha_3 \alpha_4 = \zeta \, \underbrace{\alpha_1 \alpha_3} \, \underbrace{\alpha_2 \alpha_4} \qquad (2)$$

In practise, many contractions are zero (see 19.6, 19.7):

Examples:

$$\langle c_1^\dagger c_2 c_3^\dagger c_4 \rangle_0 = \langle c_1^\dagger c_2 \rangle_0 \langle c_3^\dagger c_4 \rangle_0 + \langle c_1^\dagger c_4 \rangle_0 \langle c_2 c_3^\dagger \rangle_0 \qquad (3)$$

$$\langle c_1^\dagger c_2^\dagger c_3 c_4 \rangle_0 = \zeta \langle c_1^\dagger c_3 \rangle_0 \langle c_2^\dagger c_4 \rangle_0 + \langle c_1^\dagger c_4 \rangle_0 \langle c_2^\dagger c_3 \rangle_0 \qquad (4)$$

<u>Wick's theorem for thermal GF's:</u> $\quad c_j \equiv c_{\lambda_j}(\tau_j)$

$$\mathcal{G}_0^{(n)}(1,\dots,n;1',\dots,n') \equiv (-)^n \langle T_\tau c_1 \dots c_n c^\dagger_{n'} \dots c_{1'} \rangle_0 = \underset{\text{all permutations}}{\sum} \zeta^P \mathcal{G}_0^{(1)}(1,P(1)) \dots \mathcal{G}_0^{(1)}(n,P(n)) \tag{1}$$

$$\text{where} \quad \mathcal{G}_0^{(1)}(i,j) = -\langle T_\tau c_i c_j^\dagger \rangle_0 = -\Big(\underbrace{\theta(\tau_i-\tau_j)}_{\equiv \theta_{ij}} \langle c_i c_j^\dagger \rangle_0 + \zeta\, \theta(\tau_j - \tau_i) \langle c_j^\dagger c_i \rangle_0 \Big) \tag{2}$$

Useful notation:
$$\mathcal{G}_0^{(1)}(i,j) \overset{(19.4,5)}{=} -\Big(\theta_{ij}\, \underbrace{c_i c_j^\dagger} + \zeta\, \theta_{ji}\, \underbrace{c_j^\dagger c_i}\Big) \equiv -T_\tau \underbrace{c_i c_j^\dagger} \tag{3}$$

<u>Example:</u> consider $\tau_{2'} > \tau_1 > \tau_{1'} > \tau_2$:

$$\mathcal{G}_0^{(2)}(1,2;1',2') = (-)^2 \langle T\, c_1 c_2 c^\dagger_{2'} c^\dagger_{1'} \rangle_0 \overset{\text{action of } T_\tau:}{=} \zeta (-)^2 \langle c^\dagger_{2'} c_1 c^\dagger_{1'} c_2 \rangle_0 \tag{4}$$

ζ, ζ^2

$$\overset{\text{Wick (21.1)}}{=} \zeta (-)^2 \Big[\underbrace{c_1 c^\dagger_{1'}}\, \underbrace{c^\dagger_{2'} c_2} + \underbrace{c^\dagger_{2'} c_1}\, \underbrace{c^\dagger_{1'} c_2} \Big] \tag{5}$$
$$= (-)^2 T_\tau \big[c_1 c_2 c^\dagger_{2'} c^\dagger_{1'} \big] \tag{7}$$

reexpress in terms of $\mathcal{G}_0$, using (3): $\qquad$ OR : rewrite in terms of time-ordered contractions:

$$= \cancel{\zeta} \Big[\mathcal{G}_0^{(1)}(1,1')\, \cancel{\zeta}\, \mathcal{G}_0^{(1)}(2,2') + \zeta\, \mathcal{G}_0^{(1)}(1,2')\, \cancel{\zeta}\, \mathcal{G}_0^{(1)}(2,1') \Big] \tag{6}$$
$$= (-)^2 \Big[\big(T_\tau \underbrace{c_1 c^\dagger_{1'}}\big)\big(T_\tau \underbrace{c_2 c^\dagger_{2'}}\big) + \zeta \big(T_\tau c_1 c^\dagger_{2'}\big)\big(T_\tau c_2 c^\dagger_{1'}\big)\Big] \tag{8}$$

$= (1)$ ✓

<u>General proof:</u>

$$\mathcal{G}_0^{(n)}(\tau_1,\dots,\tau_n;\tau_{1'},\dots\tau_{n'}) = (-)^n \langle T_\tau \big(c_1 c_2 \dots c_n c^\dagger_{n'} \dots c^\dagger_{2'} c^\dagger_{1'}\big) \rangle_0$$

for any given time ordering, stands for correspondingly ordered product of operators (with an extra sign $\zeta^{P'}$, where $P' = 0$ or 1 for even/odd number of permutations needed relative to the reference order, $\tau_1 > \tau_2 > \dots \tau_n > \tau_{n'} > \dots > \tau_{2'} > \tau_{1'}$)

For any such order, we can apply Wick's theorem :

$$\overset{\text{(as in 22.7)}}{=} \overset{(21.1)}{} (-)^n\, T_\tau \sum_P \zeta^P \big(c_1 c_2 \dots c_n c_{P(n')} \dots c^\dagger_{P(2')} c^\dagger_{P(1')} \big) \tag{4}$$

T_τ acting on product of contractions produces extra $\zeta^{P'}$

[Sum over all permutations P relative to reference order generates all nonzero pairwise contractions, including signs]

reorder, producing only even powers of ζ

$\zeta^{2(n-1)} \cdot \zeta^{2(n-2)} \dots = (\text{even power of } \zeta) = 1$ (3)

then "factorize" action of T_τ on contractions:

$$\overset{\text{(as in 22.8)}}{=} \sum_P \zeta^P (-)\big(T_\tau \underbrace{c_1 c^\dagger_{P(1')}}\big) (-) \big(T_\tau \underbrace{c_2 c^\dagger_{P(2')}}\big) \dots (-)\big(T_\tau \underbrace{c_n c^\dagger_{P(n')}}\big) \tag{5}$$

$$\overset{\text{use (22.2)}}{=} \sum_P \zeta^P\, \mathcal{G}_0^{(1)}(1,P(1'))\, \mathcal{G}_0^{(1)}(2,P(2')) \dots \mathcal{G}_0^{(1)}(n,P(n')) \overset{\checkmark}{=} (22.1) \quad \square \tag{6}$$

Comment: Wick's theorem also holds for linear combinations of creation and annihilation operators. In particular for field operators: PT24

$$\hat\psi(x) \equiv \sum_\lambda \varphi_\lambda(x)\, c_\lambda \qquad \hat\psi^\dagger(x) \equiv \sum_\lambda \varphi^*_\lambda(x)\, c^\dagger_\lambda \quad : \tag{1}$$

Define free correlators (exp. value taken with $\hat H' = 0$),

$$\mathcal{G}_0^{(1)}(1,1') \equiv (-)\langle T_\tau\, \hat\psi(1)\, \hat\psi^\dagger(1')\rangle_0 \tag{2}$$

$$\mathcal{G}_0^{(n)}(1,2,\dots,n;1',2',\dots,n') \equiv (-)^n \langle T_\tau\, \hat\psi(1)\dots\hat\psi(n)\,\hat\psi^\dagger(n')\dots\hat\psi^\dagger(1')\rangle_0 \tag{3}$$

$$= \sum_{\lambda_1}\dots\sum_{\lambda_n}\sum_{\lambda_{n'}}\dots\sum_{\lambda_{1'}} \varphi_{\lambda_1}(x_1)\dots\varphi_{\lambda_n}(x_n)\,\varphi^*_{n'}(x_{n'})\dots\varphi_{1'}(x_{1'})$$
$$\underbrace{\times(-)^n\langle T_\tau\, c_{\lambda_1}(\tau_1)\dots c_{\lambda_n}(\tau_n)\, c^\dagger_{\lambda_{n'}}(\tau_{n'})\dots c^\dagger_{\lambda_{1'}}(\tau_{1'})\rangle_0}_{(22.1)}$$
$$\sum_P \xi^P\, (-)\langle T_\tau\, c_{\lambda_1}(\tau_1)\, c^\dagger_{\lambda_{P(1)}}(\tau_{P(1)})\rangle_0 \dots (-)\langle T_\tau\, c_{\lambda_n}(\tau_n)\, c^\dagger_{\lambda_{n'}}(\tau_{n'})\rangle \tag{4}$$

$$= \sum_P \xi^P\, \mathcal{G}_0^{(1)}(1,P(1'))\dots\mathcal{G}_0^{(1)}(n,P(n')) \tag{5}$$

Wick's theorem — alternative proof, using equations of motion

PT25

$$\text{Suppose}\quad \hat H_0 = \int dx\, \hat\psi^\dagger(x)\, h_0(x)\, \hat\psi(x) = \text{quadratic} \tag{1}$$

$$\Rightarrow\quad -\partial_{\tau_1}\hat\psi(1) \overset{\text{compare (6.6)}}{=} h_0(1)\,\hat\psi(1) \tag{2}$$

$$\mathcal{G}_0^{(n)}(1,2,\dots,n;\underbrace{1',2',\dots,n'}) \equiv (-)^n\langle T_\tau\,\hat\psi(1)\dots\hat\psi(n)\,\hat\psi^\dagger(n')\dots\psi(1')\rangle_0 \tag{3}$$

(sometimes we will write these indices as $n+1,\dots,2n$)

$$\left[-\partial_{\tau_1} - h_0(1)\right]\mathcal{G}_0^{(n)}(1,\dots,n;1',\dots,n') \overset{(2)}{=} 0 + (-)^{n-1}\,\partial'_{\tau_1}\langle T_\tau\,\hat\psi(1)\dots\hat\psi(n)\,\psi^\dagger(n')\dots\psi^\dagger(1')\rangle \tag{4}$$

prime means: acts only on time ordering θ's

will be shown below, see (28.4)
$$= \sum_{j'=1'}^{n'} \xi^{j'-1}\,\delta(1,j')\,\mathcal{G}_0^{(n-1)}(2,\dots n;1',\dots\cancel{j'}\dots n') \tag{5}$$

(5) is diff.eq. for $\mathcal{G}_0^{(n)}$ in the variables τ_1, x_1, whose solution can be found using:

$$\left(-\partial_{\tau_1} - h_0(1)\right)\mathcal{G}_0^{(1)}(1,1') = \delta(1,1'). \tag{6}$$

PT 26

namely:

$$\mathcal{G}_0^{(n)}(1,\dots,n;1',\dots,n') = \sum_{j'=1}^{n'} \zeta^{j'-1}\, \mathcal{G}_0^{(1)}(1,j')\, \mathcal{G}_0^{(n-1)}(2,\dots,n;1',\dots\cancel{j'}\dots n') \qquad (1)$$

This is a recursion relation for $\mathcal{G}^{(n)}$ i.t.o. $\mathcal{G}^{(n-1)}$:

$$\mathcal{G}_0^{(2)}(1,2;1',2') = \mathcal{G}_0^{(1)}(1,1')\,\mathcal{G}_0^{(1)}(2,2') + \zeta\, \mathcal{G}_0^{(1)}(1,2')\,\mathcal{G}_0^{(1)}(2,1') \qquad (2)$$

(compare 22.6)

General solution:

$$\mathcal{G}_0^{(n)}(1,\dots,n;1',\dots,n') = \sum_{\text{all permutations}} \zeta^P\, \mathcal{G}_0^{(1)}(1,P(1))\dots\mathcal{G}_0^{(1)}(n,P(n)) \qquad (3)$$

(in agreement with 24.5)

Note: for fermions, this can be expressed as a determinant:

$$(\text{if } \zeta=-1) = \begin{vmatrix} \mathcal{G}_0(1,1') & \mathcal{G}_0(1,2') & \cdots & \mathcal{G}_0(1,n') \\ \mathcal{G}_0(2,1') & & & \vdots \\ \vdots & & & \vdots \\ \mathcal{G}_0(n,1') & & & \mathcal{G}_0(n,n') \end{vmatrix} \qquad (4)$$

PT 27

Proof of $(25.5)_{RHS}$: T_τ implies a sum over all possible time orderings, multiplied by θ-functions,

($\hat{A}_j$ stands for $\psi(j)$ or $\psi^\dagger(j)$)

$$T_\tau(\underbrace{\hat{A}_1}_{\text{"}\hat{\psi}(1)\text{"}} \hat{A}_2 \dots \hat{A}_j \dots \hat{A}_{2n}) = \sum_P \zeta^P\, \theta(\tau_{P(1)} > \tau_{P(2)} \dots > \tau_{P(2n)})\, \hat{A}_{P(1)} \hat{A}_{P(2)} \dots \hat{A}_{P(2n)} \qquad (1)$$

ζ^{j-2}

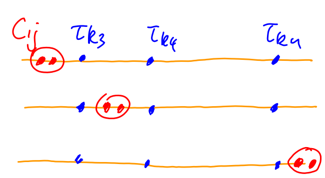
- For a given index $j \neq 1$ (i.e. $j \in \{2,\dots,2n\}$) consider all terms in (1) with A_1 and $\hat{A}_j$ next to each other, i.e. containing the combination:

$$\hat{C}_{1j} \equiv \left(\theta_{1j}\, \hat{A}_1 \hat{A}_j + \zeta\, \theta_{j1}\, \hat{A}_j \hat{A}_1\right) \qquad (3)$$

with an arbitrary but fixed order of the other $2n-2$ operators, labelled $k_3,\dots,k_{2n}$:

$$\hat{S}_{1j} \equiv \zeta^{j-2} \begin{bmatrix} \hat{C}_{1j}\, \hat{A}_{k_3} \hat{A}_{k_4} \dots \hat{A}_{k_{2n}}\, \theta(\tau_1,\tau_j > \tau_{k_3}) \\ + \hat{A}_{k_3}\, \hat{C}_{1j}\, \hat{A}_{k_4} \dots \hat{A}_{k_{2n}}\, \theta(\tau_{k_3} > \tau_1,\tau_j > \tau_{k_4}) \\ + \dots \\ + \hat{A}_{k_3} \hat{A}_{k_4} \dots \hat{A}_{k_{2n}}\, \hat{C}_{1j}\, \theta(\tau_{k_4} > \tau_1,\tau_j) \end{bmatrix} \zeta^{P'}\, \theta(\tau_{k_3} > \tau_{k_4} > \dots \tau_{k_{2n}}) \qquad (4)$$

sign needed to move $\hat{A}_j$ just to the right of $\hat{A}_1$



Then

$$T_\tau(\hat{A}_1 \dots \hat{A}_{2n}) = \sum_{j \neq 1} \underbrace{T_{(\tau_{k_3}\dots\tau_{k_{2n}})}}_{\text{time ordering of "other times"}} \left(\hat{S}_{1j}\right) \qquad (5)$$

$\boxed{PT28}$

Now:

since $\hat A_1 = \hat\psi_1$

$$\partial'_{\tau_1} \hat C_{1j} = \delta(\tau_1-\tau_j)\big(\underbrace{\hat A_1\hat A_j - \xi \hat A_j \hat A_1}_{[\hat A_1,\hat A_j]_\xi}\big) = \begin{cases}\delta(1,j) & \text{if } \hat A_j = \psi_j^\dagger \\ 0 & \text{if } \hat A_j = \psi_{j'}\end{cases} \equiv \underbrace{A_1 A_j}_{\times} \qquad (1)$$

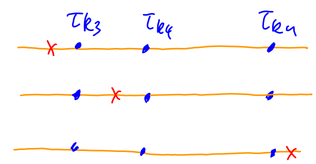
∂_{τ_1} $\odot$

So, when we consider the action of ∂_{τ_1} only on the θ-functions in $\hat C_{1j}$ (its action on other θ-functions, containing $\theta(\tau_1,\tau_j>\tau_{k_3})$ to $\theta(\tau_{k_4}>\tau_1,\tau_j)$, will be considered later, when treating different choices for j), we obtain

acting only on C_{ij}'s

so: $$\partial'_{\tau_1}\hat S_{1j} \overset{(26.4)}{=} \xi^{j-2}\, \underbrace{A_1 A_j}\, A_{k3}\dots A_{kn} \qquad (2)$$

$$\times\underbrace{\Big[\theta(\tau_1>\tau_{k_3}) + \theta(\tau_{k_4}>\tau_1>\tau_{k_3}) + \dots + \theta(\tau_{k_{2n}}>\tau_1)\Big]}_{=\,1,\ \text{since it includes all possible orderings of } \tau_1=\tau_j \text{ relative to other times.}}$$



$\boxed{PT29}$

Sum this over all possible orderings of the "other times" τ_{k_3} to τ_{k_n}, as in (26.5):

$$\partial'_{\tau_1} T_\tau(\hat A_1\dots\hat A_{2n}) \overset{(27.5)}{=} \sum_{j\neq 1} T_{(\tau_{k3}\dots\tau_{kn})}\, \partial'_{\tau_1}(\hat S_{1j}) \overset{(2)}{=}$$

does not occur

$$= \sum_{j\neq 1} \xi^{j-2}\, \underbrace{A_1 A_j}\; T_\tau\big[\hat A_2\dots\hat A_{j-1}\, \cancel{\hat A_j}\, \hat A_{j+1}\dots\hat A_{2n}\big] \qquad (1)$$

Applied to RHS of (25.4) we get: (only non-zero terms are for $A_j \in \psi^\dagger_{j'}$)

$$(-)^{n-1}\,\partial'_{\tau_1}\langle T_\tau\, \hat\psi(1)\dots\psi(n)\,\psi^\dagger(n')\dots\psi^\dagger(1')\rangle \qquad (2)$$

$j'-1$ even

$$j = n-1+j' \quad = (-)^{n-1}\sum_{j'=2'}^{n'} \xi^{j'-1}\, \underbrace{\psi(1)\,\psi^\dagger(j')}_{(27.1)\ =\ \delta(1,j')}\, \langle T_\tau\, \hat\psi(2)\dots\hat\psi(j)\dots\hat\psi(n)\,\psi^\dagger(n')\dots\cancel{\psi^\dagger(j')}\dots\psi^\dagger(1')\rangle \qquad (3)$$

$$= \sum_{j'=2'}^{n'} \xi^{j'-1}\,\delta(1,j')\, \mathcal{G}_0^{(n-1)}(2,\dots;1',\dots\cancel{j'}\dots n') \qquad (4)$$

This is what we used in (25.5) □.