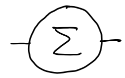
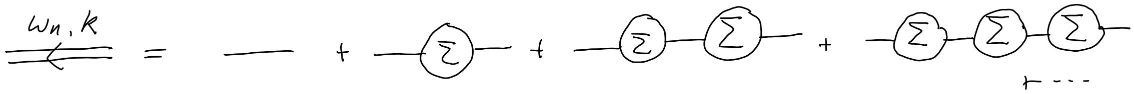
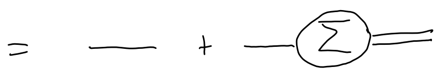
Dyson Equation

To sum contributions from a particular type of "self-energy" diagram to all orders, set up a Dyson equation





(1)



(similar to PT7.6) (2)

in (ω_n, k) space these are simple products

full $\bar{\mathcal{G}}$, must be found self-consistently

$$\bar{\mathcal{G}}(i\omega_n, k) = \bar{\mathcal{G}}_0(i\omega_n, k) + \bar{\mathcal{G}}_0(i\omega_n, k) \cdot \Sigma(i\omega_n, k) \cdot \bar{\mathcal{G}}(i\omega_n, k) \tag{3}$$

Solve: $$\bar{\mathcal{G}}(1 - \mathcal{G}_0 \Sigma) = \mathcal{G}_0 \tag{4}$$

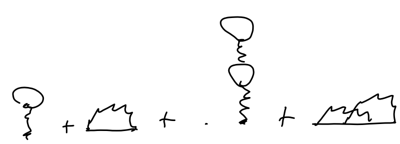
$$\mathcal{G}(i\omega_n, k) = \frac{\mathcal{G}_0}{1 - \mathcal{G}_0 \Sigma} = \frac{1}{\mathcal{G}_0^{-1} - \Sigma} = \frac{1}{i\omega_n - \varepsilon_k - \Sigma(i\omega_n, k)} \tag{5}$$

Analytic continuation to find $G^{R/A}$:

$$G^{R/A}(\omega, k) = \mathcal{G}(i\omega_n \to \omega \pm i\alpha, k) = \frac{1}{\omega \pm i\alpha - \varepsilon_k - \Sigma^{R/A}(\omega, k)} \tag{1}$$

where $$\Sigma^{R/A}(\omega, k) = \Sigma(i\omega_n \to \omega \pm i\alpha, \varepsilon_k) \tag{2}$$

Note: pert. expansion not for $\mathcal{G}$, but for Σ =



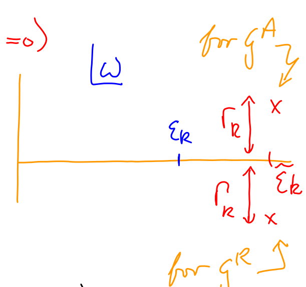
(3)

This allows infinite sets of diagrams for $\mathcal{G}$ to be summed!

(reduces to $\varepsilon_k \mp i\alpha$ if $\Sigma = 0$)

Poles of $G^{R/A}$: at $\omega^* \equiv \tilde{\varepsilon}_k \mp i\Gamma_k$ (4)

satisfying: $\omega^* \overset{(1)}{=} \varepsilon_k + \Sigma^R(\omega^*, k)$ (5)



If $\Gamma_k \ll \tilde{\varepsilon}_k$: $$\tilde{\varepsilon}_k = \varepsilon_k + \mathrm{Re}\, \Sigma^R(\tilde{\varepsilon}_k \pm i\alpha, k) \tag{6}$$

(simplify $\tilde{\omega} = \tilde{\varepsilon}_k \cancel{\pm i\Gamma_k}$ on RHS)

$$\Gamma_k = -\mathrm{Im}\, \Sigma(\tilde{\varepsilon}_k \pm i\alpha, k) \tag{7}$$

Expand denominator of $G^{R/A}$ in (53.1) about pole: $\omega = \omega^* + \underbrace{(\omega - \omega^*)}_{= \text{small}}$ (1)

$\boxed{PT54}$

$$\omega \pm i\alpha - \varepsilon_k - \Sigma^{R/A}(\omega,k) = \underbrace{\omega^* - \varepsilon_k - \Sigma^{R/A}(\omega^*,k)}_{(53.5)\ = 0} + (\omega-\omega^*)\underbrace{\left[1 - \frac{\partial \Sigma^{R/A}(\omega,k)}{\partial \omega}\Big|_{\omega^*}\right]}_{\equiv 1/a_k \quad (3)} \qquad (2)$$

negligible compared to $i\,\mathrm{Im}(\omega^*) = \mp i \Gamma_k$

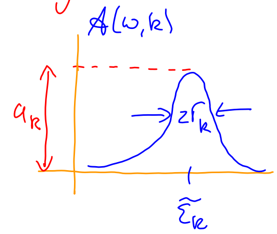
$[0,1] \ni a_k \equiv$ "quasi-particle weight"

$\tilde{\varepsilon}_k \equiv$ "renormalized energy"

$\Gamma_k \equiv$ decay rate $= \frac{1}{\text{lifetime}}$.

Then,
$$G^{R/A}(\omega,k) \overset{(53.1)}{=} \frac{a_k}{\omega - \tilde{\varepsilon}_k \pm i\Gamma_k} \qquad (4)$$

$$A(\omega,k) = -\frac{1}{\pi}\,\mathrm{Im}\,G^R(\omega,k) = a_k \frac{\Gamma_k/\pi}{(\omega-\tilde{\varepsilon}_k)^2 + \Gamma_k^2} \qquad (5)$$



$$G^{R/A}(t,k) \overset{(GF50.7)}{=} a_k\,\theta(\pm t)\,e^{-i\tilde{\varepsilon}_k t}\,e^{-\Gamma_k t} \qquad (6)$$

So, the effect of $\Sigma^R(\omega,k)$ on pole directly shows up in time-dependence of response function.

If $\tilde{\varepsilon}_k \gg \Gamma_k$ (well-defined peak in $A(\omega,k)$, many oscillations before $G(t,k)$ decays), we say: "quasi-particles are well-defined".

Hartree-Fock approximation

$\boxed{PT55}$

$$H = H_0 + H' \qquad (1)$$

$$H_0 = \int dx_1\, \psi^\dagger(x_1)\, h_0(x_1)\, \psi(x_1) \qquad (2)$$

$$H' = \frac{1}{2}\int dx_1'' \int dx_1'\; \psi^\dagger(x_1'')\, \psi^\dagger(x_1')\, v(x_1'',x_1')\, \psi(x_1')\, \psi(x_1'') \qquad (3)$$

(all times equal)

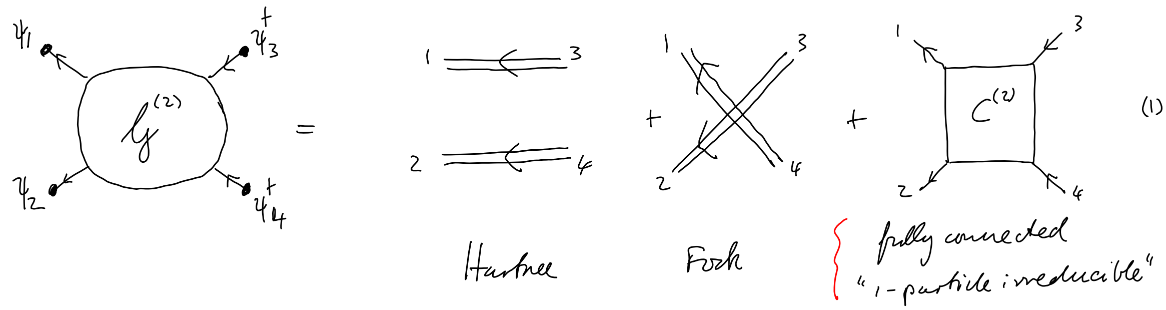
$$\mathcal{G}(1,2) = -\langle T_\tau\, \psi(1)\, \psi^\dagger(2)\rangle \qquad (4)$$

$$-\partial_{\tau_1}\psi(1) = \underbrace{[\psi(1), H]}_{\text{equal times}} = h_0(1)\,\psi(1) + \int dx_1'\, \psi^\dagger(1')\, v(x_1,x_1')\, \psi(1')\, \psi(1) \qquad (5)$$

$$(-\partial_{\tau_1} - h_0)\,\mathcal{G}(1,2) = \delta(1,2) + \int dx_1'\, v(x_1,x_1')(-)\underbrace{\langle T_\tau\, \psi^\dagger(1')\, \psi(1')\, \psi(1)\, \psi^\dagger(2)\rangle}_{\zeta \langle T_\tau\, \psi(1)\psi(1')\, \psi^\dagger(1')\, \psi^\dagger(2)\rangle} \qquad (6)$$

$$= \delta(1,2) - \zeta \int dx_1'\, v(x_1,x_1')\, \mathcal{G}(1,1';2,1') \qquad (7)$$

General structure of two-point function: $\mathcal{G}(1,2;3,4) = (-)^2\langle T_\tau \psi_1 \psi_2 \psi_4^\dagger \psi_3^\dagger \rangle$ $\boxed{PT56}$



(1)

$$\mathcal{G}^{(2)}(1,2;3,4) = \mathcal{G}(1,3)\,\mathcal{G}(2,4) + \zeta\,\mathcal{G}(1,4)\,\mathcal{G}(2,3) + C^{2}(1,2;3,4) \qquad (2)$$

$C^{(2)}(1,2;3,4)$, defined by Eq.(2), capture correlations between particles (i.e. those part of propagation that do not factorize.

For small density, 2-particle correlations are weak $\Rightarrow$ neglect C_2.

$(56.2)\big|_{C_2=0}$ into (55.7): $\mathcal{G}(1,1';2,1')$ $\boxed{PT57}$

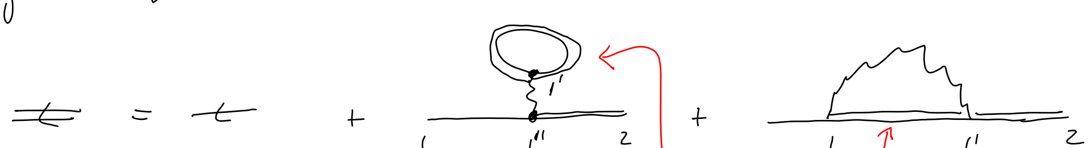
$$\left(-\partial_{\tau_1} - h_0\right)\mathcal{G}(1,2) = \delta(1,2) - \zeta\int dx_1'\, v(x_1,x_1')\left[\mathcal{G}(1,2)\,\mathcal{G}(1',1'_+) + \zeta\,\mathcal{G}(1,1'_+)\,\mathcal{G}(1',2)\right] \qquad (1)$$

Hartree - Fock eq.

but, $\left(-\partial_{\tau_1} - h_0(1)\right)\mathcal{G}_0(1,2) = \delta(1,2)$ (2)

$\Rightarrow$ solution for (1): (3)

$$\mathcal{G}(1,2) = \mathcal{G}_0(1,2) - \zeta\int d1''\int d1'\, \mathcal{G}_0(1,1'')\, v(1'',1')\left[\mathcal{G}(1'',2)\,\mathcal{G}(1',1'_+) + \zeta\,\mathcal{G}(1'',1'_+)\,\mathcal{G}(1',2)\right]$$



(4)

self-consistent Hartree - Fock! full GF occurs on RH side!

Note: (3) could directly have been written down from diagrams in (4), using Feynman rules.