After impurity averaging, translational invariance is restored: $\mathcal{G}^X(1,2) = \mathcal{G}^X(1-2)$: $\boxed{\text{Dis 33}}$

$$\langle \mathcal{G}^{R/A}(\omega;1,2') \rangle_{imp} = \underbrace{\frac{1}{Vol}\sum_{\vec{k}}}_{\int (dk)} e^{i\vec{k}(\vec{r}_1 - \vec{r}_2{}')} \mathcal{G}^{R/A}(\omega,\vec{k}) \qquad (1)$$

$$\vec{j}_{11'} \overset{(27.3)}{=} \frac{-ie}{2m}(\vec{\nabla}_1 - \vec{\nabla}_{1'}) \Rightarrow$$

$$j^{\alpha}_{11'} j^{\beta}_{22'} \langle \mathcal{G}^R(\omega;1,2') \rangle_{imp} = \text{'' \quad '' \quad ''} \quad \frac{ek^{\alpha}}{2m}\frac{ek^{\beta}}{2m} \qquad (2)$$

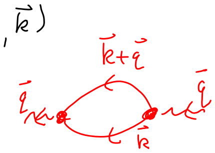
$$\mathcal{G}_{\alpha\beta}(i\omega_m;1,2) \overset{(30.2,4)}{=} j^{\alpha}_{11'} j^{\beta}_{22'} \frac{1}{\beta}\sum_{\omega_n} \bar{\mathcal{G}}(i\omega_m + i\omega_n;1,2')\, \bar{\mathcal{G}}(i\omega_n,2,1') \qquad (3)$$

$$= \frac{1}{\beta}\sum_{\omega_n}\int (dk')\int (dk)\; e^{i\vec{k}'(\vec{r}_1 - \vec{r}_2{}')} e^{i\vec{k}(\vec{r}_2 - \vec{r}_1{}')} \bar{\mathcal{G}}(i\omega_m + i\omega_n,\vec{k}')\bar{\mathcal{G}}(i\omega_n,\vec{k}) \frac{e}{2m}(\vec{k}'+\vec{k})^{\alpha}\frac{e}{2m}(\vec{k}+\vec{k}')^{\beta} \qquad (4)$$

$$\mathcal{G}_{\alpha\beta}(i\omega_m,\vec{q}) = \int d\vec{r}_{12}\, e^{-i\vec{q}\cdot(\vec{r}_1 - \vec{r}_2)} \mathcal{G}_{\alpha\beta}(i\omega_m,\vec{q}) \quad \Rightarrow \quad \vec{k}' = \vec{k}+\vec{q} \qquad (5)$$

$$= \frac{1}{\beta}\sum_{\omega_m} e^2 \left(\frac{2\vec{k}+\vec{q}}{2m}\right)^{\alpha}\left(\frac{2\vec{k}+\vec{q}}{2m}\right)^{\beta} \bar{\mathcal{G}}(i\omega_m + i\omega_n,\vec{k}+\vec{q})\,\bar{\mathcal{G}}(i\omega_n,\vec{k}) \qquad (6)$$

Current-current correlator has vectors associated with its vertices



After performing Matsubara sums as on p.31, we can write, in analogy to (32.0): $\boxed{\text{Dis 34}}$

$$\mathcal{G}^R_{\alpha\beta}(\omega,\vec{q}) = \mathcal{G}^{RA}_{\alpha\beta}(\omega,\vec{q}) + \mathcal{G}^{RR+AA}_{\alpha\beta}(\omega,\vec{q}) \qquad (1)$$

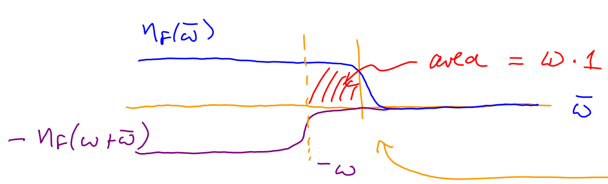
will cancel diamagnetic term in (29.4).

as in (33.6)

$$\mathcal{G}^{RA}_{\alpha\beta}(\omega,\vec{q}) \overset{(32.1)}{=} -\frac{i}{2\pi}\underbrace{\int d\bar{\omega}\,\{n_F(\bar{\omega}) - n_F(\omega+\bar{\omega})\}}_{\equiv I(\omega)} \int (dk)\, e^2 \left(\frac{2\vec{k}+\vec{q}}{2m}\right)^{\alpha}\left(\frac{2\vec{k}+\vec{q}}{2m}\right)^{\beta} \underbrace{\mathcal{G}^R(\omega+\bar{\omega},\vec{k}+\vec{q})\,\mathcal{G}^A(\bar{\omega},\vec{k})}_{\left[\frac{1}{\omega+\bar{\omega}-\varepsilon_{\vec{k}+\vec{q}}+i/2\tau}\;\frac{1}{\bar{\omega}-\varepsilon_{\vec{k}}-i/2\tau}\right]} \qquad (2)$$

$I(\omega) = \omega$, if rest of integrand does not depend on $\bar{\omega}$ (3)

ε_k-integral converges, since integrand $\sim 1/\varepsilon_k^2$ (4)



(2) is nonzero only if $\quad -\omega \lesssim \bar{\omega} \leq 0$. (due to) (5)

Usual assumptions about applied field: $\quad q \ll \frac{1}{\ell}$, (6)

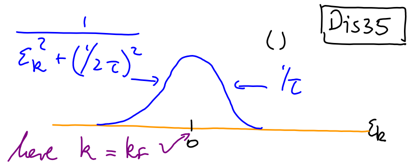
$|\omega| \ll \frac{1}{\tau}$ (7)

(field changes slowly on length and time scales that characterize impurity scattering)

Dis35

$(34.5),(34.7)$: $\quad \Rightarrow \quad |\bar\omega| \ll \frac{1}{\tau}$ (1)

$$\Rightarrow \quad |\varepsilon_k| \lesssim \frac{1}{\tau} \ll \varepsilon_F , \qquad (2)$$



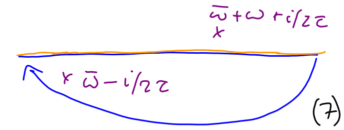
$$\Rightarrow \quad \left|\frac{k^2}{2m} - \varepsilon_F\right| \simeq 0 \quad \Rightarrow \quad |k| \simeq |k_F| \qquad (3)$$

$$\overset{(34.6)}{\Rightarrow} \quad |q| \ll |k| \qquad (4)$$

$$(2k+q)^\alpha \simeq 2k_F^\alpha \qquad (5)$$

$$\varepsilon_{\vec k+\vec q} = \frac{(\vec k+\vec q)^2}{2m} - \varepsilon_F = \varepsilon_k + \frac{1}{m}\vec k\cdot\vec q + \underbrace{q^2/2m}_{\overset{(4)}{\text{neglect}}} \qquad (6)$$

Exploiting these inequalities, we get:



$$\mathcal{G}^{RA}_{\alpha\beta}(\omega,\vec q) \overset{(34.2)}{=} I(\omega)\, e^2 \underbrace{\frac{\mathcal{N}(0)}{m^2}}_{\frac{\bar n}{m}\frac{3}{k_F^2}} \int \frac{d\Omega_{k_F}}{4\pi} k_F^\alpha k_F^\beta \int \frac{d\varepsilon_k}{2\pi i} \frac{1}{\varepsilon_k + \frac{\vec k_F\cdot\vec q}{m} - \omega - \bar\omega - i/2\tau}\, \frac{1}{\varepsilon_k - \bar\omega + i/2\tau} \qquad (7)$$

(14.3),(14.6)

$$= \omega\, e^2 \frac{\bar n}{m} \underbrace{\int \frac{d\Omega_{k_F}}{4\pi} \frac{3 k_F^\alpha k_F^\beta}{k_F^2}}_{= \delta_{\alpha\beta} \text{ if } \vec q = 0} \frac{1}{\omega - \frac{\vec k_F\cdot\vec q}{m} + i/\tau} \qquad (8)$$

$\bar\omega$-dependence drops out, so $I(\omega) = \omega$

Dis36

Insert (35.8) into first term of (29.4):

$$\underset{\text{RA-contribution}}{\sigma_{\alpha\beta}(\omega,\vec q)} = -\left(\frac{2}{i\omega}\right) \mathcal{G}^{RA}_{\alpha\beta}(\omega,\vec q) \qquad (1)$$

$$= \frac{2e^2 \bar n \tau}{m} \int \frac{d\Omega_{k_F}}{4\pi} \frac{3 k_F^\alpha k_F^\beta}{k_F^2} \frac{1}{1 - i\omega\tau + i\tau \vec k_F\cdot\vec q/m} \qquad (2)$$

This implies a non-local relation between current and vector potential:

$$\delta\vec J(\omega,\vec q) \overset{(29.3)}{=} \frac{3e^2 \overset{\text{total density}}{(2\bar n)}}{m} \int \frac{d\Omega_{k_F}}{4\pi} \frac{\vec k_F(\vec k_F\cdot \vec A(\omega,\vec q))}{k_F^2} \frac{i\omega\tau}{1 - i\omega\tau + i\tau \vec k_F\cdot\vec q/m} \qquad (3)$$

$\vec E(\vec q,\omega)\tau$

This result can also be derived from Boltzmann eq. approach, using the "relaxation-time approximation".

It can be used to explain the "anomalous skin effect" (dissipation of current occurs within "skin depth" of surface; skin depth decreases with increasing ω).

Simple limits:

$$q \to 0: \quad \sigma_{\alpha\beta}(\omega, q=0) = \delta_{\alpha\beta} \underbrace{\frac{2e^2 \bar{n} \tau}{m}}_{\equiv \sigma_0 \,\equiv\, \text{Drude conductivity}} \frac{1}{1 - i\omega\tau} \tag{1}$$

$$\sigma_0 = \frac{2e^2 \bar{n}\tau}{m} = 2e^2 \tau \frac{N(0) k_F^2}{3 m^2} = 2e^2 N(0) \underbrace{\frac{v_F^2 \tau}{3}}_{\equiv \text{"diffusion constant"}} = 2e^2 N(0) D \tag{2}$$

(1) is the Drude result, also obtainable via Boltzmann equation.

Our derivation required only $\frac{1}{k_F l} \ll 1$ and $\frac{1}{\tau \varepsilon_F} \ll 1$.

For Boltzmann derivation, it is unclear whether one ever needs $\frac{1}{\tau T} \ll 1$ or not. Our derivation shows: no assumptions on T are required.

Cancellation of G^{RR+AA} and diamagnetic term for $\omega \to 0, q \to 0$

The following identity holds exactly: (for proof, see p. 39):

$$\int d\vec{r}_2 \; j^\alpha_{11'} \, j^\beta_{22'} \underbrace{\mathcal{G}^{R/A}_{12'}(\omega)\, \mathcal{G}^{R/A}_{21'}(\omega)}_{\equiv \mathcal{G}^{R/A}(\omega; 1, 2')} = -ie \, \underbrace{j^\alpha_{11'}}_{-\frac{ie}{2m}(\partial_{r_1} - \partial_{r_1'})^\alpha} \underbrace{r^\beta_{11'}}_{(r_1 - r_{1'})^\beta} \mathcal{G}^{R/A}_{11'}(\omega) = -\delta_{\alpha\beta} \frac{e^2}{m} \mathcal{G}^{R/A}_{11'}(\omega) \tag{1}$$

Inserting this into (29.2):

$$\sigma^{RR+AA}_{\alpha\beta}\begin{pmatrix}\omega \to 0 \\ q = 0\end{pmatrix} = -\frac{2}{i\omega}\left\{\frac{e^2}{m} \bar{n}\, \delta_{\alpha\beta} + \frac{i}{2\pi} \int d\bar{\omega}\, n_F(\bar{\omega}) \int d\vec{r}_2 \left[\mathcal{G}^R_{12'}(\bar{\omega})\, \mathcal{G}^R_{21'}(\bar{\omega}) - \mathcal{G}^A_{12'}(\bar{\omega})\, \mathcal{G}^A_{21'}(\bar{\omega})\right]\right\} \tag{2}$$

(32.2) with $\omega = 0$

Fourier backtransformation, with $\vec{q} = 0$

$$\overset{(1)}{=} -2 \frac{\delta_{\alpha\beta}}{i\omega} \frac{e^2}{m} \left\{ \bar{n} - \int d\bar{\omega}\, n_F(\bar{\omega}) \underbrace{\frac{i}{2\pi}\left[\mathcal{G}^R_{11'} - \mathcal{G}^A_{11'}\right](\bar{\omega})}_{A_{11'}(\bar{\omega}) \,=\, -\frac{i}{\pi} \operatorname{Im} \mathcal{G}^R_{11'}(\omega)} \right. \tag{3}$$

$$= 0 \tag{4}$$

$$\text{since} \quad \int d\bar{\omega}\, n_F(\bar{\omega})\, A_{11'}(\bar{\omega}) = \bar{n}(\vec{r}_1) = \bar{n} = \text{constant} \tag{5}$$

Proof of identity (38.1) $\boxed{\text{Dis39}}$

Let $\hat{H}(\hat{\vec{p}},\hat{\vec{r}}) = \frac{\hat{p}^2}{2m} + U_{imp}(\vec{r})$ (1)

be the single-particle Hamiltonian for a given disorder realization,

with exact eigenstates $|\lambda\rangle$: $\hat{H}(\hat{\vec{p}},\hat{\vec{r}})|\lambda\rangle = \xi_\lambda |\lambda\rangle$ (2)

and eigenfunctions $\langle \vec{r}|\lambda\rangle = \psi_\lambda(\vec{r})$ (3)

exact retarded/advanced GF: $\mathcal{G}^{R/A}_{ij}(\omega) \overset{(PT59.5)}{=} \sum_\lambda \frac{\psi_\lambda(\vec{r}_i)\psi^*_\lambda(\vec{r}_j)}{\omega - \xi_\lambda \pm i0^+}$ (4)

$$= \sum_\lambda \langle \vec{r}_i|\lambda\rangle \frac{1}{\omega - \xi_\lambda \pm i0^+}\langle\lambda|\vec{r}_j\rangle = \langle \vec{r}_i| \frac{1}{\omega - \hat{H} \pm i0^+}|\vec{r}_j\rangle \quad (5)$$

Now consider gauge-transformation, with spatially uniform $\vec{A}$-field:

$$\hat{H}(\hat{\vec{p}},\hat{\vec{r}}) \longrightarrow \hat{\tilde{H}}(\hat{\vec{p}},\hat{\vec{r}}) = \hat{H}(\hat{\vec{p}} + e\vec{A},\vec{r}) = \hat{H}(\vec{p},\vec{r}) + \vec{A}\cdot\hat{\vec{j}} + \frac{e^2}{2m}\vec{A}^2 \quad (6)$$

$\boxed{\text{Dis40}}$

$$\psi_\lambda(\vec{r}) \to \tilde{\psi}_\lambda(\vec{r}) \equiv e^{-ie\vec{A}\cdot\vec{r}}\psi_\lambda(\vec{r}) = \langle\vec{r}|\tilde{\lambda}\rangle \quad (1)$$

Then $\hat{\tilde{H}}|\tilde{\lambda}\rangle = \xi_\lambda|\tilde{\lambda}\rangle$ (2)

↖ same eigenvalues as in (39.2)

$$\left[\text{since } (\underbrace{\hat{\vec{p}}}_{-i\vec{\nabla}} + e\vec{A})e^{-ie\vec{A}\cdot\vec{r}}\psi_\lambda(\vec{r}) = e^{-ie\vec{A}\cdot\vec{r}}(\hat{\vec{p}} - e\vec{A} + e\vec{A})\psi_\lambda(\vec{r})\right]$$

and $\mathcal{G}^{R/A}_{ij}(\omega) \longrightarrow \tilde{\mathcal{G}}^{R/A}_{ij}(\omega) \overset{(1)\text{ in }(39.4)}{=} e^{-ie\vec{A}\cdot(\vec{r}_i - \vec{r}_j)}\mathcal{G}^{R/A}_{ij}(\omega)$ (3)

$$\overset{(1)}{=} \sum_\lambda \frac{\tilde{\psi}_\lambda(\vec{r}_i)\tilde{\psi}^*_\lambda(\vec{r}_j)}{\omega - \xi_\lambda \pm i0^+} = \sum_\lambda \langle\vec{r}_i|\tilde{\lambda}\rangle\frac{1}{\omega - \xi_\lambda \pm i0^+}\langle\tilde{\lambda}|\vec{r}_j\rangle \quad (4)$$

$$\overset{(2)}{=} \langle\vec{r}_i|\frac{1}{\omega - \hat{\tilde{H}} \pm i0^+}|\vec{r}_j\rangle \quad (5)$$

expand (3) and (5) to linear order in $\vec{A}$, using (39.6) for expanding $\hat{\tilde{H}}$.

$$\frac{1}{x-y} = \frac{1}{x} + \frac{1}{x}y\frac{1}{x} + \dots$$

$\boxed{\text{Dis41}}$

$$-ie\vec{A}\cdot(\vec{r}_i-\vec{r}_j)\, G^{R/A}_{ij}(\omega) = \sum_{\lambda\lambda'} \langle \vec{r}_i|\lambda\rangle\langle\lambda| \frac{1}{\omega-\hat{H}\pm i0^+}\, \vec{A}\cdot\hat{\vec{j}}\, \frac{1}{\omega-\hat{H}\pm i0^+} |\lambda'\rangle\langle\lambda'|\vec{r}_j\rangle \tag{1}$$

$$= \sum_{\lambda\lambda'} \psi_\lambda(\vec{r}_i)\, \frac{1}{\omega-\xi_\lambda\pm i0^+}\, \underbrace{\langle\lambda|\vec{A}\cdot\hat{\vec{j}}|\lambda'\rangle}\, \frac{1}{\omega-\xi_{\lambda'}\pm i0^+}\, \psi^*_{\lambda'}(\vec{r}_j) \tag{2}$$

matrix elements of current operator: $\vec{A}\int d\vec{r}_\ell\, \vec{j}_{\ell\ell'}\, \psi^*_\lambda(\vec{r}_{\ell'})\, \psi_{\lambda'}(\vec{r}_\ell)$ (3)

$$= \vec{A}\cdot \int d\vec{r}_\ell\, \vec{j}_{\ell\ell'}\, G^{R/A}_{i\ell'}(\omega)\, G^{R/A}_{\ell j}(\omega) \tag{4}$$

This proves the identity $\int d\vec{r}_{\ell\ell'}\, G^{R/A}_{i\ell'}(\omega)\, G^{R/A}_{\ell j}(\omega) = -ie\,\vec{r}_{ij}\, G^{R/A}_{ij}(\omega)$ (5)

used in (25.1). $\square$.