Gorkov's equations

The Hamiltonian of the interacting system

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} = \sum_{\alpha} \int d^3\vec{r}\, \hat{\psi}^+_\alpha(\vec{r}) \left[-\frac{\vec{\nabla}^2}{2m} - \mu \right] \hat{\psi}_\alpha(\vec{r}) +$$

$$+ \frac{1}{2} \sum_{\alpha\beta} \int d^3r\, U_{eff}\, \hat{\psi}^+_\alpha(\vec{r}) \hat{\psi}^+_\beta(\vec{r}) \hat{\psi}_\beta(\vec{r}) \hat{\psi}_\alpha(\vec{r})$$

where the interaction term has been written as for point-like interaction though U_{eff} must correspond to the BCS model in real calculus.

Eq/mot for the Heisenberg operators (in a real t)

$$i \frac{\partial}{\partial t} \hat{\psi}_\alpha(\vec{x}) = [\hat{\psi}_\alpha, \hat{H}] \quad // \vec{x} = \{\vec{r}, t\} //$$

+ usual anticommutation relations

$$\{ \hat{\psi}_\alpha(\vec{r}, t), \hat{\psi}^+_\beta(\vec{r}', t) \} = \delta_{\alpha\beta}\, \delta(\vec{r} - \vec{r}')$$

$$\{ \hat{\psi}_\alpha(r, t), \hat{\psi}_\beta(\vec{r}', t) \} = \{ \hat{\psi}^+_\alpha(r, t), \hat{\psi}^+_\beta(\vec{r}', t) \} = 0$$

Calculating commutator with the kinetic part (at g_{eff})

$$i \frac{\partial}{\partial t} \hat{\psi}_\alpha(t, \vec{r}) \underset{\lambda=0}{=} \sum_{\alpha'} \int d^3\vec{r}' \Big[\hat{\psi}_\alpha(\vec{r}) \hat{\psi}^+_{\alpha'}(\vec{r}') \underbrace{\left[-\frac{\vec{\nabla}^2}{2m} - \mu \right]}_{\hat{\varepsilon}} \hat{\psi}_{\alpha'}(\vec{r}')$$

$$- \hat{\psi}^+_{\alpha'}(\vec{r}') \hat{\varepsilon}\, \hat{\psi}_{\alpha'}(\vec{r}') \hat{\psi}_\alpha(\vec{r}) \Big] = \sum_{\alpha'} \int d^3\vec{r}' \{ \hat{\psi}_\alpha(\vec{r}), \hat{\psi}^+_{\alpha'}(\vec{r}') \}$$

$\Rightarrow -1$

$$\hat{\varepsilon}\, \hat{\psi}_{\alpha'}(\vec{r}') \} = -\left(\frac{\vec{\nabla}^2}{2m} + \mu \right) \hat{\psi}_\alpha(t, \vec{r})$$

and similar for $\partial_t \hat{\psi}^+_\alpha$.

Eq. of motion for $\hat{\psi}$ and $\hat{\psi}^+$

$$\begin{cases} i\dfrac{\partial\hat{\psi}_\alpha}{\partial t} = -\left(\dfrac{\nabla^2}{2m}+\mu\right)\hat{\psi}_\alpha - \sum_\gamma \lambda \hat{\psi}^+_\gamma \hat{\psi}_\gamma \hat{\psi}_\alpha \\ i\dfrac{\partial\hat{\psi}^+_\alpha}{\partial t} = +\left(\dfrac{\nabla^2}{2m}+\mu\right)\hat{\psi}^+_\alpha + \sum_\gamma \lambda \hat{\psi}^+_\alpha \hat{\psi}^+_\gamma \hat{\psi}_\gamma \end{cases}$$

Let's use them to derive eq. of motion for the Green's functions

$$i\frac{\partial}{\partial t_1} G_{\alpha\beta} = \langle \hat{T}_t \partial_{t_1}\hat{\psi}_\alpha(x_1)\hat{\psi}^+_\beta(x_2)\rangle + 1\, \delta_{\alpha\beta}\delta(x_1-x_2)$$

where the 2nd term follows from a discontinuity of G at $t_1 \to t_2$:

$$G_{\alpha\beta}|_{t_1=t_2+0} - G_{\alpha\beta}|_{t_1=t_2-0} = -i\langle\{\psi_\alpha(x_1)\psi^+_\beta(x_2)\}\rangle =$$

$$= -i\,\delta_{\alpha\beta}\,\delta(\vec{r}_1-\vec{r}_2)$$ || $\langle\ldots\rangle$ over give state, for example, over GS ||

Using eq. of motion for ψ-operators we find and assuming translational invariance

$$\left(i\partial_t + \frac{\nabla^2}{2m}+\mu\right)G_{\alpha\beta}(X-X') - i\lambda\langle \hat{T}_t \underbrace{\hat{\psi}^+_\gamma\hat{\psi}_\gamma\hat{\psi}_\alpha}_{X}\underbrace{\hat{\psi}^+_\beta}_{X'}\rangle =$$

$$= \delta_{\alpha\beta}\,\delta(X-X')$$

Consider the ψ^4 term and apply the Wick theorem

$$\langle N|\hat{T}_t \hat{\psi}^+_\gamma \hat{\psi}_\gamma \hat{\psi}_\alpha \hat{\psi}^+_\beta|N\rangle \approx -G_{\alpha\gamma}(0)\,G_{\gamma\beta}(X-X') + G_{\gamma\gamma}(0)\times$$

(1) (1) ↑ similar to free particles

$$\times G_{\alpha\beta}(X-X') + \langle N|\hat{T}\hat{\psi}_\gamma(X)\psi_\alpha(X)|N+2\rangle\langle N+2|\hat{T}\hat{\psi}^+_\gamma(X)\hat{\psi}^+_\beta(X')|N\rangle$$

We neglect renormalization of excitation spectrum due to interactions and account only for the anomalous expectation values in N-Me which describe the condensate of the Cooper-pairs

$$\Rightarrow \langle N|\hat{T}_t\, \hat{\psi}^+_\gamma(x)\hat{\psi}_\gamma(x)\,\hat{\psi}_\alpha(x)\,\hat{\psi}_\beta(x')|N\rangle \to$$

$$- F_{\gamma\alpha}(0)\, F^+_{\gamma\beta}(x-x') = \text{// separate out spin part} //$$

$$= -\,\delta_{\alpha\beta}\; F(0)\, F^+(x-x')$$

Denote $\Xi(x) = i\,F(x,x)$; $\Xi^*(x) = -i\,F^+(x,x)$.

Ξ is just a const in a stationary ~~and~~ system which does not move (at a macro-level).

Now we rewrite the eq/motion for G: (the spin factor $\delta_{\alpha\beta}$ can be canceled out)

$$\left(i\partial_t + \frac{\nabla^2}{2m} + \mu \right) G(x-x') + \lambda\, \Xi\, F^+(x-x') = \delta(x-x')$$

Normal G is coupled to anomalous F.

Important we have taken into account only appearance of the condensate due to the interactions (the BCS reduced model)

The 2nd eq/motion for F is needed to get a closed system

Calculations are very similar to G:

$$\left(i\frac{\partial}{\partial t} - \left[\frac{\vec{\nabla}^2}{2m} + \mu\right]\right) F^+(x-x') + \lambda\, \Xi^* G(x-x') = 0$$

check it yourselves!

Note, that $\delta(x-x')$ does not appear in the 2nd equation because F & F^+ are continuous functions of time.

In energy - momentum space:

$$\begin{cases} (\varepsilon - \xi(\vec{p}))\, G(\vec{p},\varepsilon) + \lambda\, \Xi\, F^+(\vec{p},\varepsilon) = 1 \\ (\varepsilon + \xi(\vec{p}))\, F^+(\vec{p},\varepsilon) + \lambda\, \Xi^*\, G(\vec{p},\varepsilon) = 0 \end{cases}$$

where $\xi(\vec{p}) = \frac{p^2}{2m} - \mu$. ^ The Gorkov's Equations!

The same calculations in imaginary time yield

$$\begin{cases} (i\omega - \xi(\vec{p}))\, G(i\omega,\vec{p}) + \Delta F^+(i\omega,\vec{p}) = 1 \\ \Delta^* G(i\omega,\vec{p}) + (i\omega + \xi(\vec{p}))\, F^+(i\omega,\vec{p}) = 0 \end{cases}$$

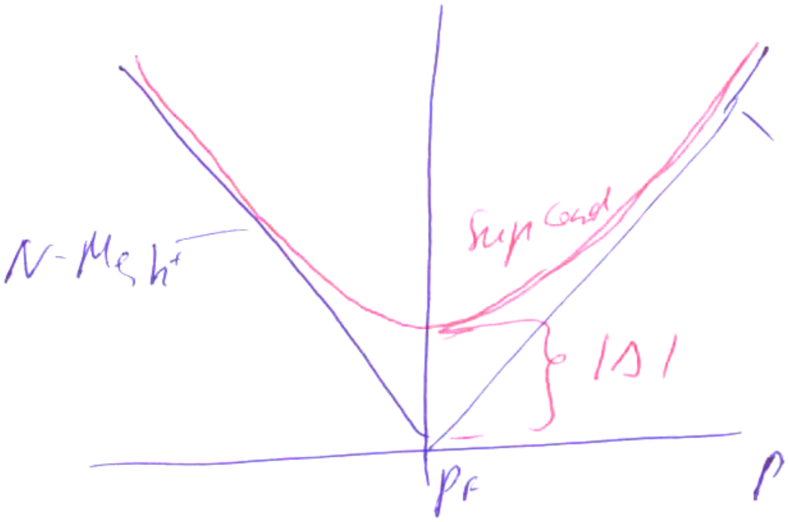
where $\Delta \equiv |\lambda|\, \Xi$

Solution: $G = -\frac{i\omega + \xi(\vec{p})}{\omega^2 + \xi^2 + |\Delta|^2}$; $F^+ = \frac{\Delta^*}{\omega^2 + \xi^2 + |\Delta|^2}$

Pole (after the analyt. cont.)

$$\omega = \sqrt{|\Delta|^2 + \xi^2(\vec{p})}$$

– dispersion relation of excitation in the Supercond.

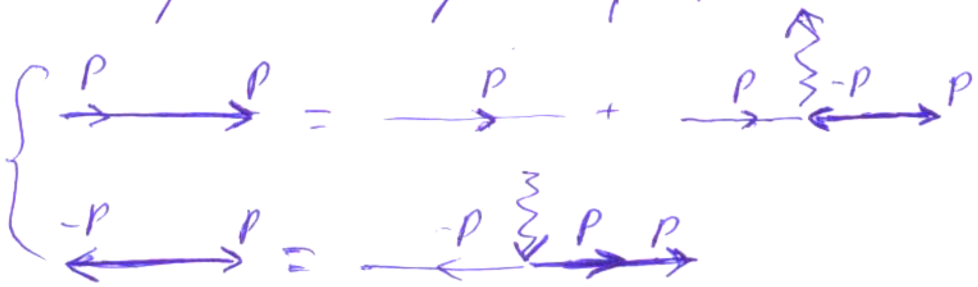


(cf. the Bogolubov solution)

Meaning of $|\Delta|$ - it's the gap of the spectrum of the (Bogolubov) excitations in the superconductor

Meaning of F - it's proportional to a wave function of the Cooper pair (in the condensate).

Graphical repr. of the Gorkov equations:



where ⟷ denotes F^+

⇡ - coherent condensate

Check yourselves that these graphs are equivalent to the analytical expression of the Gorkov's equations