Nambu matrix Green's functions

Nambu

Eqs. for the Green's functions in a supercond. can be written in a convenient matrix form.

Define

$$\hat{G}(x_1, x_2) = \begin{pmatrix} -\langle \hat{T}_\tau \psi_\alpha(x_1)\psi^+_\beta(x_2)\rangle & -\langle \hat{T}_\tau \psi_\alpha(x_1)\psi_\beta(x_2)\rangle \\ -\langle \hat{T}_\tau \psi^+_\alpha(x_1)\psi^+_\beta(x_2)\rangle & -\langle \hat{T}_\tau \psi^+_\alpha(x_1)\psi_\beta(x_2)\rangle \end{pmatrix}$$

This 2×2 matrix Green's function can be also defined via the Nambu spinors

$$\tilde{\psi}(x_1) = \begin{pmatrix} \psi_\alpha(x_1) \\ \psi^+_\beta(x_1) \end{pmatrix}; \quad \tilde{\psi}^+(x_2) = \left(\psi^+_\alpha(x_2), \psi_\beta(x_2)\right)$$

// accounts for the property $G_{\alpha\beta} \propto \delta_{\alpha\beta}$ //

$$\Rightarrow \quad \hat{G}(x_1, x_2) = -\langle \hat{T}_\tau \tilde{\psi}(x_1)\tilde{\psi}^+(x_2)\rangle$$

It can be shown that $\hat{G}$ obeys the following eq.:

$$\left(i\varepsilon_n \mathbb{1} - \hat{\tau}_z \xi(\vec{p}) - \frac{1}{2}\left(\Delta\hat{\tau}^+ + \Delta^*\hat{\tau}^-\right)\right)\hat{G}(i\varepsilon_n, \vec{p}) = \mathbb{1}$$

where $\hat{\tau}_{x,y,z}$ Pauli matrices in the Nambu space

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \hat{\tau}^\pm = \hat{\tau}_x \pm i\hat{\tau}_y$$

Nambu formalism allows to write compact eqs and to draw simpler diagrams. – automatic arrows orientation

Example: $\vec{H}$ can be taken into accoun via the vector potential by a substitution

$$\hat{\vec{p}} \to \hat{\vec{p}} - \frac{e}{c}\tau_z \vec{A}$$

DOS in the BCS model

Let's consider DoS of excitations in a Sup Cond

$$N(E) \overset{def}{\equiv} \sum_{\sigma} \sum_{\vec p} \delta(E - \varepsilon_{\sigma}(\vec p)) = 2 \sum_{\vec p} \delta(E - \varepsilon(\vec p))$$

$$\delta = -\frac{1}{\pi}\, \mathrm{Im}\, G^{(R)}$$ || recall $G^{(R)}_{(\omega,\vec p)} = \frac{1}{\omega - \xi(\vec p) + i0}$ ||

$$\Rightarrow N(E) = -\frac{2}{\pi} \sum_{\vec p} G^{(R)}(E, \vec p)$$

Solution of the Gorkov's equations

$$G(i\omega_n, \vec p) = -\frac{i\omega_n + \xi(\vec p)}{\omega_n^2 + \xi^2(\vec p) + |\Delta|^2}$$, after analyt. continuation we find

$$\Rightarrow G^{(R)}(\omega, \vec p) = -\frac{\omega + \xi(\vec p)}{-\omega_+^2 + \xi^2(\vec p) + |\Delta|^2} = \frac{\omega + \xi(\vec p)}{\omega_+^2 - \xi^2(\vec p) - |\Delta|^2}$$

$G^{(R)}$ does not depend on angles $\Rightarrow$

$$N(E)\Big|_{Vol\to\infty} = -\frac{2}{\pi} \int \frac{d\Omega_{\xi}}{4\pi} \int d\xi_p\, \nu(\xi_p)\, \mathrm{Im}\, G^R(E, \xi_p) =$$

$$= -\frac{2}{\pi}\, \mathrm{Im} \int d\xi_p\, \nu(\xi_p)\, G^R(E, \xi_p)\,, \quad \xi = \varepsilon - \mu$$

main contribution comes from $\xi \sim E^2 - |\Delta|^2$ – small

$\Rightarrow \varepsilon \sim \mu \Rightarrow$

$$N(E) \simeq -\frac{2}{\pi} \nu(\varepsilon_F)\, \mathrm{Im} \int_{-\infty}^{\infty} d\xi\, \frac{E_+ + \cancel{\xi}}{E_+^2 - \xi^2 - |\Delta|^2}$$

$E_+ = E + i0$ || $\nu(\varepsilon_F) = \frac{p_F m}{2\pi^2 \hbar^3}$ ||

Note that $\operatorname{Im} G^{(R)}(E) = \frac{1}{2i}\left(G^{(R)}(E) - G^{(A)}(E)\right)$

but $G^{(A)}(E) = \dfrac{E - i0}{(E-i0)^2 - \xi^2 - |\Delta|^2} = -\dfrac{-E+i0}{(-E+i0)^2 - \xi^2 - |\Delta|^2} =$

$$= -G^{(R)}(-E)$$

$\Rightarrow \operatorname{Im} G^{(R)}(E) = \frac{1}{2i}\left(G^{(R)}(E) + G^{(R)}(-E)\right)$ – even function of E. We can rewrite the integral

$$N(E) = \frac{2}{\pi}\,\nu(\xi_F)\,|E| \operatorname{Im}\int_{-\infty}^{\infty} \frac{d\xi}{\xi^2 + |\Delta|^2 - (E+i0)^2}$$

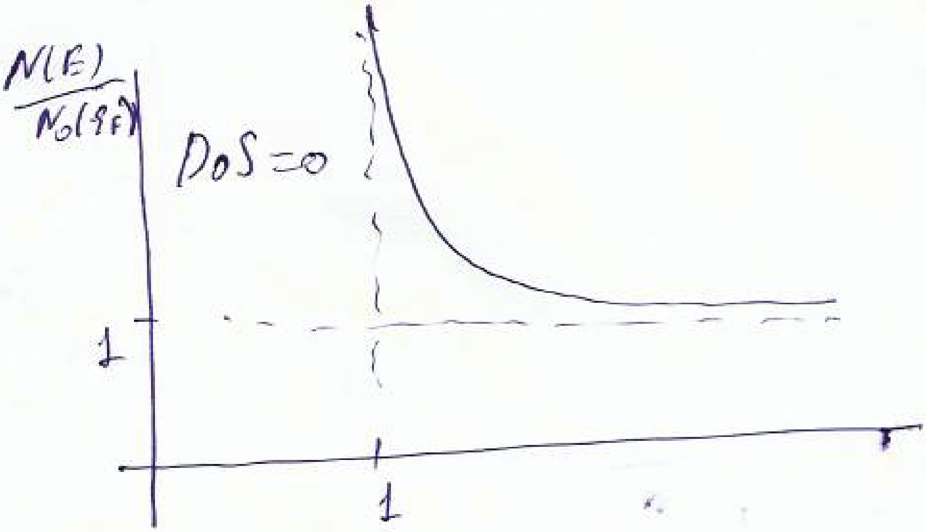
Poles $\xi = \pm\sqrt{(E+i0)^2 - |\Delta|^2} = \pm\left(\sqrt{E^2 - |\Delta|^2} + i0\right)$

Calculating the pole integral

$$N(E) = \overbrace{\frac{2}{\cancel{\pi}}\,\nu(\xi_F)}^{N_0(\xi_F)\text{ - full DoS}}\,|E|\,\cancel{2\pi}\,\operatorname{Im}\frac{1}{\cancel{2}\sqrt{|\Delta|^2 - \xi^2}}$$

which can be rewritten as

$$\frac{N(E)}{N_0(\xi_F)} = \begin{cases} \dfrac{|E|}{\sqrt{E^2 - |\Delta|^2}}, & |E| > |\Delta| \\ 0, & \text{otherwise} \end{cases}$$



$\dfrac{N(E)}{N_0(\xi_F)} = 1$ if $|E| > \omega_D$

– states are expelled when the gap is opened

The Gorkov's eq. for the gap, $\Delta(T)$

Definition $\quad \Delta = |\lambda| \Xi$, where $\Xi = F(x,x)$

Diagram:

$\Delta =$ 

where $U_{eff} = \lambda\, \delta(x - x')$ or in momentum space $U_{eff} = \lambda\, \omega_p$

Thus we can write down equation for the gap (neglecting spin indices)

$$\boxed{\Delta = |\lambda| \int \frac{d\vec{p}}{(2\pi)^3}\, T \sum_{\omega_n} F(i\omega_n; \vec{p})\, \omega_p}$$

Solution of the Gorkov's equation

$$F(i\omega_n, \vec{p}) = \frac{\Delta}{\omega_n^2 + \xi^2_{(p)} + |\Delta|^2}$$

Δ is assumed to be const in the BCS theory if the system is stationary and there is no supercurrent.

$$\Rightarrow \quad 1 = \frac{|\lambda|}{(2\pi)^3}\, T \int d^3p \sum_{\omega_n} \frac{\omega_{\vec{p}}}{\omega_n^2 + \xi^2(p) + \Delta^2} \leftarrow \text{just } |\Delta|$$

similar to the DoS

$$1 = |\lambda|\, \nu_0 \int_{-\omega_D}^{\omega_D} d\xi\, T \sum_{\omega_n} \frac{1}{\underbrace{\omega_n^2 + \xi^2 + \Delta^2}_{E^2}}$$

ω_D – comes from $\omega_{\vec{p}}$

$$T \sum_{\omega_n} \frac{1}{\omega_n^2 + E^2} = \frac{1}{2}\, \frac{\tanh\left(\frac{E}{2T}\right)}{E}$$

$$\Rightarrow \quad 1 = \frac{|\lambda| N_0}{2} \int_0^{\omega_D} d\xi \, \frac{\tanh\left(\frac{E(\xi)}{2T}\right)}{E(\xi)} \; ; \quad E \equiv \sqrt{\xi^2 + \Delta^2} \qquad \text{Δ-2}$$

Analysis

(a) To calculat T_c, put $\Delta = 0 \Rightarrow$

$$1 = \frac{|\lambda| N_0}{2} \int_0^{\omega_D} d\xi \, \frac{\tanh\left(\overset{x}{\xi / 2T}\right)}{\xi} = \frac{\lambda N_0}{2} \int_0^{\frac{\omega_D}{2T}} dx \, \frac{\tanh(x)}{x} -$$

– the same eq., as we obtained from the Cooper instability

$$T_c \Big|_{\substack{\text{weak}\\ \text{coupling}}} \simeq \frac{2\gamma}{\pi} \omega_D \exp\left(-\frac{1}{\lambda \nu_F}\right)$$

(b.) $T \to 0$ limit, let's find $\Delta(T=0) \equiv \Delta_0$

$$\tanh\left(\frac{E(\xi)}{2T}\right) \to 1$$

$$\Rightarrow \quad 1 = |\lambda| \nu_F \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}} = |\lambda| \nu_F \, \mathrm{arcsinh}\left(\frac{\omega_D}{\Delta_0}\right)$$

weak coupling limit

$$\log\left(2 \frac{\omega_D}{\Delta_0}\right) \simeq \frac{1}{|\lambda| \nu_F} \; , \quad \Delta_0 \simeq 2\omega_D \exp\left(-\frac{1}{\lambda \nu_F}\right)$$

$$\Rightarrow \quad \boxed{\Delta_0 \simeq \frac{\pi}{\gamma} T_c}$$ – one of the universal relations, which can be obtained from BCS.

(c) Small T, $T \ll T_c$

Let's add and subtract a part, which corresponds to $T \to 0$

$$\Rightarrow -\frac{1}{|\lambda| V_\uparrow} = \int_0^{\hbar\omega_D} d\xi \, \frac{1-\tanh\left(\frac{E(\xi)}{2T}\right)}{E(\xi)} - \underbrace{\int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\Delta^2+\xi^2}}}_{\simeq \log\left(\frac{2\omega_D}{\Delta(T)}\right)}$$ (Δ-3)

$-\frac{1}{|\lambda| V_\uparrow} = \log\left(\frac{\Delta_0}{2\omega_D}\right) \Rightarrow$ we can rewrite

$$-\frac{1}{|\lambda| V_\uparrow} + \log\left(\frac{2\omega_D}{\Delta(T)}\right) = \log\left(\frac{\Delta_0}{2\omega_D}\right) + \log\left(\frac{2\omega_D}{\Delta(T)}\right) =$$

$$= \log\left(\frac{\Delta_0}{\Delta(T)}\right)$$

To find leading correction to Δ_0 we

1) $\hbar\omega_D \to \infty$ in the 1st integral (fast convergence)

2) substitute Δ_0 for $\Delta(T)$

$$\Rightarrow \log\left(\frac{\Delta_0}{\Delta(T)}\right) \simeq \int_0^\infty d\xi \, \frac{1-\tanh\left(\frac{\sqrt{\xi^2+\Delta_0^2}}{2T}\right)}{\sqrt{\xi^2+\Delta_0^2}} =$$

$$= 2\int_0^\infty d\xi \, \frac{n_F\left(\frac{\sqrt{\xi^2+\Delta_0^2}}{T}\right)}{\sqrt{\xi^2+\Delta_0^2}} = 2\int_0^\infty dx \, \frac{n_F\left(\sqrt{x^2+\left(\frac{\Delta_0}{T}\right)^2}\right)}{\sqrt{x^2+\left(\frac{\Delta_0}{T}\right)^2}}$$

Calculating the integral, we find

$$\log\left(\frac{\Delta_0}{\Delta(T)}\right) \simeq \sqrt{\frac{2\pi T}{\Delta_0}}\, e^{-\frac{\Delta_0}{T}} + O\left(e^{-2\frac{\Delta_0}{T}}\right)$$

$\Delta(T) \simeq \Delta_0 \exp\left(-\sqrt{\frac{2\pi T}{\Delta_0}}\, e^{-\frac{\Delta_0}{T}}\right)$ - We can Taylor-expand

$$\Delta(T)|_{T \ll T_c} \simeq \Delta_0 \Big(1 - \underbrace{\sqrt{2\pi \frac{T}{\Delta_0}}\, e^{-\frac{\Delta_0}{T}}}_{\text{exp. small deviation}}\Big)$$

(d) Similar to (c): let's add and subtract a part, which corresponds to $\Delta = 0$ (Δ≠4)

$$\Rightarrow \log \frac{T_c}{T} \simeq \int_0^\infty \left(\frac{\tanh \frac{\xi}{2T}}{\xi} - \frac{\tanh\left(\frac{\sqrt{\xi^2+\Delta^2}}{2T}\right)}{\sqrt{\xi^2+\Delta^2}} \right) d\xi$$

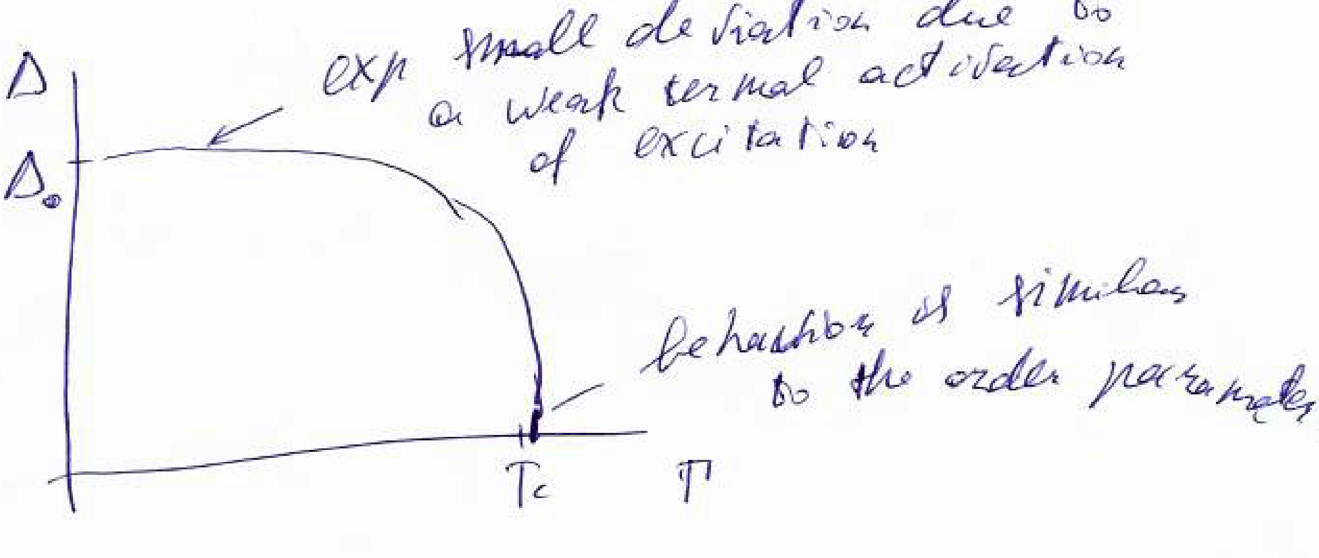
| Taylor expand in Δ

$$\log\left(\frac{T_c}{T}\right) \simeq \frac{7}{8} \frac{\zeta(3)}{\pi^2} \left(\frac{\Delta}{T}\right)^2 \Rightarrow$$

$$\Delta \simeq \pi \sqrt{\frac{8}{7\zeta(3)}}\, T \log\left(\frac{T_c}{T}\right)$$

Let's write $T \equiv T_c - \delta T$ where $\delta T \ll T_c$ and Taylor expand in $\delta T \Rightarrow$

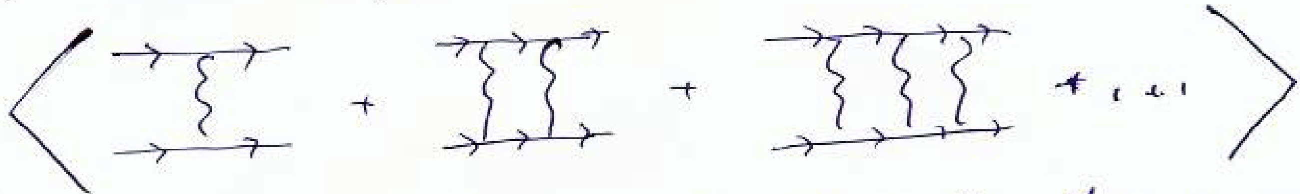
$$\Delta|_{T \to T_c} \simeq \pi \sqrt{\frac{8}{7\zeta(3)}} \sqrt{T_c(T_c - T)}$$



The Anderson theorem

Weak disorder does not destroy the Cooper pairs (if impurities are non-magnetic)

As an illustration let's consider disorder averaged Cooper ladder diagrams:



Impurity lines do not intersect each other (each ... yield $\frac{1}{k_F l}$ or $\frac{1}{\varepsilon_F \tau} \ll 1$)

All Green's functions are disorder averaged

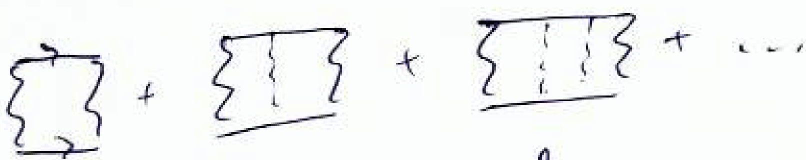
$$G(i\omega_n, p) = \frac{1}{i\omega_n - \xi(\vec{p}) + \underbrace{i\frac{\operatorname{sign}\omega_n}{2\tau}}_{\text{self energy}}}$$

Reminder

diverges at $\vec{p}_1 + \vec{p}_2 = \vec{q}$ - small

- diffusive ladder also gives the leading contr. at small q.

Let's dress each block of the Cooper ladder by additional diffusive ladder



Then $\langle \Gamma_{Cooper} \rangle = -\frac{\lambda}{1 + \lambda \Pi}$

where

$\bar{\Pi} =$ 

Each buildin block of this ladder at $\vec{p}_1 - \vec{p}_2 \to 0$

$$B(i\omega_n) = \int \frac{d^3p}{(2\pi)^3}\, G(i\omega_n, \vec{p})\, G(-i\omega_n, -\vec{p}) = \frac{2\pi\nu(F)\tau}{1+2\tau|\omega_n|}$$

(see the PS-8 for details)

Now let's sum the diffusive ladder:

$$\sum_n \underbrace{\left(\frac{1}{2\pi\nu(F)\tau}\right)}_{=\ldots}^{n-1} B^n(i\omega_n) = \frac{B(i\omega_n)}{1 - \frac{B(i\omega_n)}{2\pi\nu(F)\tau}} =$$

$$= \frac{2\pi\nu(F)\tau}{1+2\tau|\omega_n|} \frac{1}{1 - \cancel{\frac{1}{2\pi\nu(F)}} \frac{1}{1+2\tau|\omega_n|}} = \frac{\cancel{2\pi\nu(F)\tau}}{1+\cancel{2\tau|\omega_n|}} \frac{\cancel{1+2\tau|\omega_n|}}{\cancel{1+2\tau|\omega_n|-1}}$$

$$= \frac{\pi\nu(F)}{|\omega_n|}$$ – does not depend on τ!

Matsubara sum over ω_n yields the same expression for $\bar{\Pi}$ as in the clean case $\Rightarrow$ T_c ~~dead~~ is not changed due to disorder.