Lineare Gleichungssysteme und Matrizen

$$
\begin{array}{llllll}
z_1: & \frac{1}{3}x_1 & + \frac{1}{3}x_2 & + \frac{1}{3}x_3 & = 1 \\
z_2: & x_1 & + 2x_2 & + 3x_3 & = 2 \\
z_3: & 2x_1 & + 3x_2 & + x_3 & = 1
\end{array}
\Longrightarrow
\begin{array}{l}
x_1 = ? \\
x_2 = ? \\
x_3 = ?
\end{array}
$$

Gauß-Verfahren:

$$
\begin{array}{lllllll}
z_1': & 3z_1: & x_1 & + x_2 & + x_3 & = 3 \\
z_2': & z_2 - 3z_1: & \underline{\underline{0}} & + x_2 & + 2x_3 & = -1 \\
z_3': & z_3 - 6z_1: & \underline{\underline{0}} & + x_2 & - x_3 & = -5 \\
 & & & & & \\
z_1'': & z_1' - z_2': & x_1 & + 0 & - x_3 & = 4 \\
z_2'': & z_2' + 2z_3': & \underline{\underline{0}} & + 3x_2 & + \underline{\underline{0}} & = -11 \\
z_3'': & z_2' - z_3': & 0 & + 0 & + 3x_3 & = 4
\end{array}
$$

$$
\begin{array}{lllll}
z_1'' + z_3''/3: & x_1 & + 0 & + 0 & = 16/3 \\
z_2''/3: & 0 & + x_2 & + 0 & = -11/3 \\
z_3''/3: & 0 & + 0 & + x_3 & = 4/3
\end{array}
$$

Kurznotation: Erweiterte Matrix:

$$
\begin{array}{llll}
\frac{1}{3}x_1 & + \frac{1}{3}x_2 & + \frac{1}{3}x_3 & = 1 \\
x_1 & + 2x_2 & + 3x_3 & = 2 \\
2x_1 & + 3x_2 & + x_3 & = 1
\end{array}
\qquad
\begin{array}{l}
x_1 = a_1 \\
x_2 = a_2 \\
x_3 = a_3
\end{array}
\quad \Uparrow
$$

$$
\begin{array}{l}
z_1: \\ z_2: \\ z_3:
\end{array}
\left(\begin{array}{ccc|c}
1/3 & 1/3 & 1/3 & 1 \\
1 & 2 & 3 & 2 \\
2 & 3 & 1 & 1
\end{array}\right)
\Longrightarrow
\left(\begin{array}{ccc|c}
1 & 0 & 0 & a_1 \\
0 & 1 & 0 & a_2 \\
0 & 0 & 1 & a_3
\end{array}\right)
$$

$$
\begin{array}{ll}
z_1' = 3z_1: & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -5 \end{array}\right) \\
z_2' = z_2 - 3z_1: & \\
z_3' = z_3 - 6z_1: &
\end{array}
$$

$$
\begin{array}{ll}
z_1'' = z_1' - z_2': & \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 3 & 0 & -11 \\ 0 & 0 & 3 & 4 \end{array}\right) \\
z_2'' = z_2' + 2z_3': & \\
z_3'' = z_2' - z_3': &
\end{array}
$$

$$
\begin{array}{ll}
z_1''' = z_1'' + z_3''/3: & \left(\begin{array}{ccc|c} 1 & 0 & 0 & 16/3 \\ 0 & 1 & 0 & -11/3 \\ 0 & 0 & 1 & 4/3 \end{array}\right) \\
z_2''' = z_2''/3: & \\
z_3''' = z_3''/3: &
\end{array}
\Longrightarrow
\begin{array}{l}
x_1 = 16/3 \\
x_2 = -4/3 \\
x_3 = 4/3
\end{array}
$$

Matrizen:

Eine Matrix ist ein rechteckiges Zahlenschema mit m Zeilen und n Spalten:

$$
\underline{\underline{A}} =
\begin{array}{c}
\\ m \text{ Zeilen}
\end{array}
\left\{
\overbrace{
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & & A_{ij} & \\
A_{m1} & A_{m2} & \cdots & A_{mn}
\end{bmatrix}
}^{n \text{ Spalten}}
\right.
= (A_{ij})
$$

A_{ij}, $i = 1, \dots, m$

$j = 1, \dots, n$

Quadratische Matrix: $n = m$.

Eine (von unendlich vielen) Anwendung: lineare Gleichungssysteme:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

in "Matrix notation": $\underline{\underline{A}} \cdot \vec{x} = \vec{b}$,

"Multiplikation eines Spaltenvektors mit einer Matrix"

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Multiplikation Matrix - Vektor $\longrightarrow$ Vektor:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix}$$

$$b_i = \sum_{j=1}^{n} a_{ij} x_j$$

[$(m \times n)$-Matrix] $\cdot$ [Spaltenvektor mit n Einträgen]

= [Spaltenvektor mit m Einträgen]

$$\underline{\underline{A}} \cdot \vec{x} = \vec{b} \qquad (1)$$ ⑦

Matrix-Multiplikation : Matrix , Matrix → Matrix.

$$\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} = \underline{\underline{\mathbb{1}}}$$

$\underline{\underline{A}}^{-1} \cdot (1)$:
$$\underbrace{\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}}}_{\mathbb{1}} \cdot \vec{x} = \underline{\underline{A}}^{-1} \cdot \vec{b}$$

$$\underline{\underline{\mathbb{1}}} \cdot \vec{x} = \underline{\underline{A}}^{-1} \cdot \vec{b}$$

$$\vec{x} = \underline{\underline{A}}^{-1} \cdot \vec{b}$$

Matrix-Multiplikation: ⑧

Voraussetzung: $\underset{n\times m}{\underline{\underline{B}}} \cdot \underset{m\times l}{\underline{\underline{A}}} = \underset{n\times l}{\underline{\underline{C}}}$

$$\begin{pmatrix} B_{11} & \cdots & B_{1j} & \cdots & B_{1m} \\ \vdots & & \vdots & & \vdots \\ B_{i1} & \cdots & B_{ij} & \cdots & B_{im} \\ \vdots & & \vdots & & \vdots \\ B_{n1} & \cdots & B_{nj} & \cdots & B_{nm} \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1l} \\ \vdots & & & & \\ A_{j1} & \cdots & A_{jk} & \cdots & A_{jl} \\ \vdots & & & & \\ A_{m1} & \cdots & A_{mk} & \cdots & A_{ml} \end{pmatrix} = \begin{pmatrix} C_{11} & \cdots & C_{1k} & \cdots & C_{1l} \\ \vdots & & \vdots & & \vdots \\ C_{i1} & \cdots & \boxed{C_{ik}} & \cdots & C_{il} \\ \vdots & & \vdots & & \vdots \\ C_{n1} & \cdots & C_{nk} & \cdots & C_{nl} \end{pmatrix}$$

Reihe i; Spalte j; Spalte k; Reihe j; Spalte k; Reihe i

$$C_{ik} = B_{i1}A_{1k} + B_{i2}A_{2k} + \dots B_{ij}A_{jk} + \dots + B_{im}A_{mk}$$
$$= \sum_{j=1}^{m} B_{ij}A_{jk}$$

$$\underline{\underline{B}} \cdot \underline{\underline{A}} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} =$$

$\underline{\underline{B}} \setminus \underline{\underline{A}} \rightarrow$	$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 \\ & \\ 4 & 5 \\ & \\ 1 & 0 \end{pmatrix}$	$1\cdot 0 + 2\cdot 2$ $4\cdot 0 + 5\cdot 2$ $1\cdot 0 + 0\cdot 2$	$1\cdot 1 + 2\cdot 1$ $4\cdot 1 + 5\cdot 1$ $1\cdot 1 + 0\cdot 1$

$$= \begin{pmatrix} 4 & 3 \\ 10 & 9 \\ 0 & 1 \end{pmatrix} = C$$

$$\mathbb{1} = \text{Einheitsmatrix} :\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = n \times n\text{-Matrix}$$

$$\mathbb{1}_{ij} = \begin{cases} 1 & \forall\, i = j \\ 0 & \forall\, i \neq j \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 0 + 0 \\ 0 + y + 0 \\ 0 + 0 + z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underline{\underline{A}} \cdot \underline{\underline{\mathbb{1}}} = \underline{\underline{A}} \quad , \quad \mathbb{1} \cdot \underline{\underline{A}} = \underline{\underline{A}}$$

I. Allgemeinen: $\underline{\underline{A}} \cdot \underline{\underline{B}} \neq \underline{\underline{B}} \cdot \underline{\underline{A}}$ (nicht-kommutativ)

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 10 & 9 \end{pmatrix}$$

$$\underline{\underline{B}} \cdot \underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 6 & 9 \end{pmatrix}$$

Matrix-Diagonalisieren:

$$\underline{\underline{U}}\;\underline{\underline{A}}\;\underline{\underline{U}}^{-1} = \underline{\underline{D}} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

gesucht: $\underline{\underline{U}}$ gegeben: $\underline{\underline{A}}$ diagonal: $\underline{\underline{D}}$

Beispiel: Partialbruchzerlegung:

$$I = \int dx \, \frac{4x - 1}{(x+2)(x-1)^2} = \int \frac{R(x)}{Q(x)} = \int dx \left[\frac{A}{x - x_1} + \underbrace{\frac{B}{x - x_2} + \frac{C}{(x-x_2)^2}}_{\frac{\tilde{B}x + \tilde{C}}{(x-x_2)^2}} \right]$$

Nullstellen v. Q: $x_1 = -2$, $x_2 = 1$ (doppelt)

$$\frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2}$$

$$4x - 1 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

$0 \cdot x^2$

$$x^2 \underbrace{(A + B)}_{=0} \Rightarrow A = -B$$

$$= A\left[\cancel{x^2} - 2x + 1 - (\cancel{x^2} + x - 2)\right] + C(x+2)$$

$$= x\,(\underbrace{-2A \quad -A \quad +C}) + (\underbrace{A + 2A + 2C})$$

$$4 = -2A - A + C \quad \Rightarrow \quad C = 1$$

$$-1 = A + 2A + 2C \quad \Rightarrow \quad A = -1 \quad \Rightarrow \quad B = +1$$

$$I = \int dx \left[\frac{-1}{x+2} + \frac{1}{x-1} + \frac{1}{(x-1)^2} \right]$$

$$= -\ln|x+2| + \ln|x-1| - \frac{1}{(x-1)} + \text{Const.}$$

Check: $$\frac{-(x-1)^2 + (x+2)(x-1) + x+2}{(x+2)(x-1)^2} = \frac{-\cancel{x^2} + 2x - 1 + \cancel{x^2} + x - 2 + x + 2}{} \qquad \frac{4x - 1}{(\ \ )(\ \ )^2} \ \checkmark$$

Substitution

$$I := \int dx\, f(g(x))\, g'(x)$$

$$I = \int dx \underbrace{\overset{g'}{\cos x}}_{dy'} \underbrace{(\overset{g}{\sin x + b})}_{F(g(x))}{}^{c} = \int dy\, (y+b)^c$$

$$= \frac{1}{c+1} (y+b)^{c+1} = \frac{1}{c+1} (\sin x + b)^{c+1}$$

$$y = \sin(x)$$

$$\frac{dy}{dx} = \cos x \quad \Rightarrow \quad dy \text{ "=" } dx \cos x$$

Check: $$\frac{d}{dx} \frac{1}{c+1} (\sin x + b)^{c+1} = \frac{\cancel{c+1}}{\cancel{c+1}} (\sin x + b)^c \cdot \cos x \ \checkmark$$

$$I' = \int_0^{\pi/2} dx \cos x (\sin x + b)^c = \int_{y=\sin(x=0)=0}^{y=\sin(x=\pi/2)=1} dy\, (y+b)^c = \frac{1}{c+1} (y+b)^{c+1} \Big|_0^1$$

$$= \frac{1}{c+1} \left[(1+b)^c - b^{c+1} \right]$$

Alternativ: $I' = \frac{1}{c+1}(\sin x + b)^{c+1} \Big|_0^{\pi/2}$

$$= \frac{1}{c+1}\left[(1+b)^{c+1} - b^{c+1}\right]$$