Vorlesung IV, Teil B: Integration

4.10.2013 ①

Integration ist die Umkehroperation der Differenzierung

Gegeben $f(x)$:

Gesucht: "Stammfunktion" $F(x)$, für die gilt:

$$\frac{d}{dx}F(x) = F'(x) = f(x) \qquad \forall\, x \in D_f .$$

Stammfunktion ist nur bis auf eine Konstante eindeutig festgelegt.

Denn: $\frac{dF_1}{dx} = f(x)$, $\quad F_2(x) = F_1(x) + c$ ⇐ Konstante

dann gilt auch $\frac{dF_2}{dx} = f(x)$, also ist $F_2(x)$ auch eine Stammfunktion.

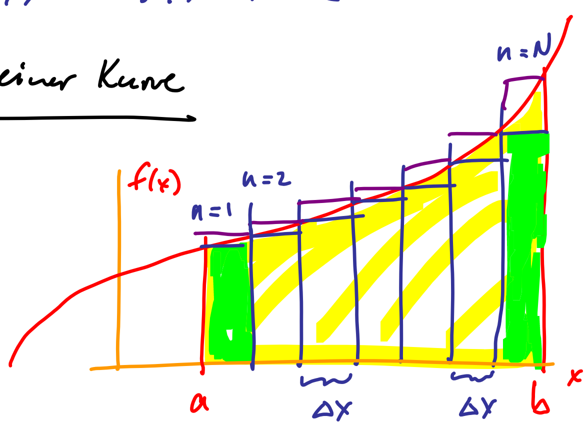
②

Menge aller Stammfunktionen einer Fkt. $f(x)$ wird durch ein "unbestimmtes Integral" ausgedrückt:

$$F(x) = \int dx\, f(x) \equiv \int f(x)\, dx$$

Zwei Stammfunktion derselben Fkt. $f(x)$ unterscheiden sich nur durch eine Konstante: $F_1(x) - F_2(x) = c$

4.1 Stammfunktion als Fläche unter einer Kurve



$f(x)$ sei stetig und monoton steigend für $x \in [a,b]$.

Gesucht: gelbe Fläche: $\equiv F$

$(b-a) = N \cdot \Delta x$

"Untersumme": $U = \left[f(a)\cdot\Delta x + f(a+\Delta x)\cdot\Delta x + \dots + f(a+(N-1)\Delta x)\cdot\Delta x \right]$ ③

"Obersumme": $O = \left[f(a+\Delta x)\cdot\Delta x + f(a+2\Delta x)\cdot\Delta x + \dots + f(\overbrace{a+N\Delta x}^{b})\cdot\Delta x \right]$

Es gilt: $O > F > U$

$$O - U = f(a+N\cdot\Delta x)\cdot\Delta x - f(a)\cdot\Delta x$$

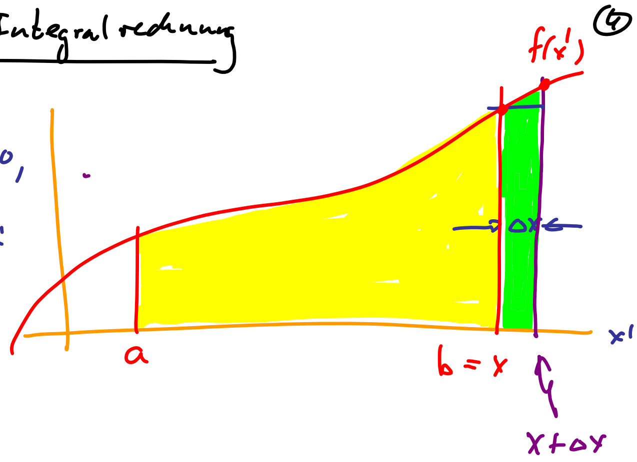
$$= \left[f(b) - f(a)\right]\cdot\Delta x \xrightarrow{\Delta x \to 0} 0$$

$$\lim_{\Delta x\to 0} U = \lim_{\Delta x \to 0} O = F$$

4.2 Hauptsatz der Differential und Integralrechnung ④

Notation: Fläche zwischen $f(x)$ und 0, mit $x \in [a,b]$, wird notiert durch:

$$\int_a^b dx\, f(x)$$



Die Zuordnung (für festes $a<b$): $F: \mathbb{R} \to \mathbb{R}$

$$b \longmapsto \int_a^b dx\, f(x) \equiv F(b)$$

definiert die sogenannte

"Flächenfunktion": $F(x) = \int_a^x dx'\, f(x')$

$(x > a)$

Behauptung: die Flächenfunktion ist eine Stammfunktion von $f(x)$.

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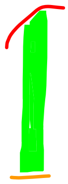
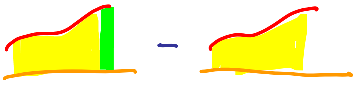
Beweisidee: $\lim_{\Delta x \to 0} \left[F(x+\Delta x) - F(x) \right] = $  $\simeq \lim_{\Delta x \to 0} \left(f(x) \cdot \Delta x \right)$

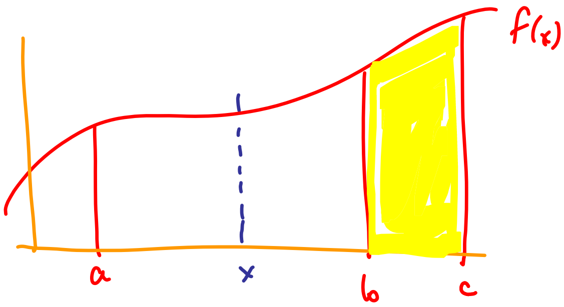


$$\lim_{\Delta x \to 0} \left[\frac{F(x+\Delta x) - F(x)}{\Delta x} \right] \simeq \lim_{\Delta x \to 0} f(x)$$

!

$F'(x) = f(x) \implies F(x)$ ist eine Stammfkt. von $f(x)$ □

$$F(x) = \int_a^x dx' \, f(x')$$



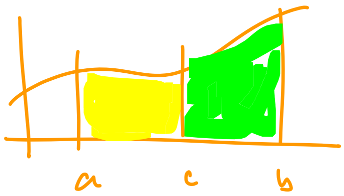
Fläche zwischen $f(x)$ und 0, für $x \in [b,c]$

$$\equiv F(x)\Big|_b^c = F(c) - F(b) = \int_b^c dx' \, f(x')$$

⑥

4.3 Eigenschaften der Integration

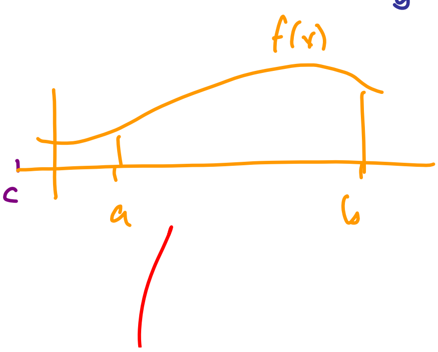
- $\int_a^b dx \, f(x) = \int_a^c dx \, f(x) + \int_c^b dx \, f(x)$

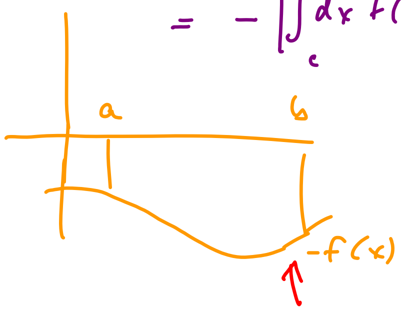


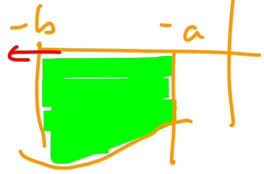
- $\int_a^a dx \, f(x) = 0$

- $\int_a^b dx \, f(x) = -\int_b^a dx \, f(x) = \int_c^b dx \, f(x) - \int_c^a dx \, f(x)$

$$= -\left[\int_c^a dx \, f(x) - \int_c^b dx \, f(x) \right]$$







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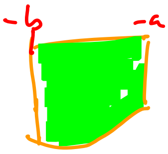
$$\int_a^b dx'\, f(x') = F(b) - F(a) = -\left(F(a) - F(b)\right)$$

$(a < b)$

$$= -\int_b^a dx'\, f(x') = \int_b^a d(-x')\, f(x')$$

lassen wir das ... $y = -x$

$$= -\int_{-b}^{-a} dy\, f(-y') = \quad = -$$



$-b < -a$

- $$\frac{d}{dx} \int_a^x dy\, f(y) = f(x)!$$

- $$\int dx \left[a\, f(x) + b\, g(x) \right] = a \int dx\, f(x) + b \int dx\, g(x)$$

- $$\int dx\, x^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + c.$$

Warum: Check: $\frac{d}{dx}\left(\frac{x^{\alpha+1}}{\alpha+1}\right) = \frac{1}{\alpha+1} (\alpha+1) \cdot x^{\alpha} = x^{\alpha}$ ✓

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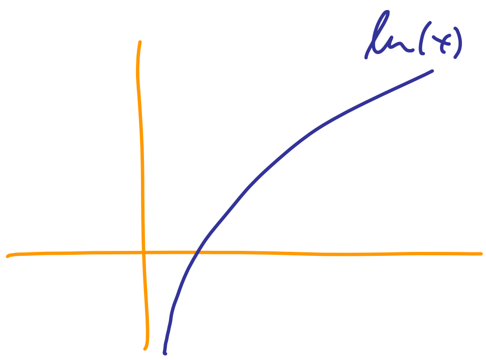
- $$\int dx\, \frac{1}{x} = \ln|x| + c$$

  weil $\frac{d}{dx} \ln(x) = \frac{1}{x}$



- $$\int dx\, e^x = e^x + c$$

  weil $\frac{d}{dx} e^x = e^x$

- $$\int dx\, \sin x = -\cos x + c$$ weil $\frac{d}{dx} \cos x = -\sin x$

- $$\int dx\, \cos x = \sin x + c$$ weil $\frac{d}{dx} \sin x = \cos x$

4.5 Partielle Integration

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u, v seien stetig differenzierbar

Produktregel: $[u(x)\, v(x)]' = u'(x) \cdot v(x) + u(x)\, v'(x)$ (1)

$\int dx\,(1)$ $(u \cdot v) = \int dx\, [u \cdot v]' = \int dx\; u' \cdot v + \int dx\; u \cdot v'$ (2)

Umstellen: $\int dx\; u \cdot v' = u \cdot v - \int dx\; u' \cdot v$ (3)

Mit Integrationsgrenzen: $\int_a^b dx\; u(x) \cdot v'(x) = \underbrace{u(x)\, v(x) \Big|_{x=a}^{x=b}}_{u(b)v(b) - u(a)v(a)} - \int_a^b dx\; u'(x)\, v(x)$ (4)

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Bsp. 1:

$I = \int_0^{\pi} dx\; \overset{u}{x} \cdot \overset{v'}{\sin x}$

$u \cdot v - \int u' \cdot v$

$v(x) = -\cos x$ | $u(x) = x$

$v'(x) = \sin x$ | $u'(x) = 1$

$= x \cdot (-\cos x) \Big|_0^{\pi} - \int_0^{\pi} dx \cdot 1 \cdot (-\cos x)$

$= \underbrace{\pi \cdot (-\cos \pi)}_{\pi} - \underbrace{0 \cdot (-\cos 0)}_{=} + \int_0^{\pi} dx\; \cos x$

$= \pi + \underbrace{\sin(x) \Big|_0^{\pi}}_{\sin \pi - \sin(0) = 0 - 0 = 0.}$

$= \pi$

(11)

Bsp. 2:

$u = x^n$, $v' = \cos x$

$$I = \int dx\; x^n \cos x$$

$v' = \cos$, $v = \sin$

$u \cdot v - \int u' \cdot v$

$$= x^n \cdot \sin x - \int dx \left(n\, x^{n-1}\right) \sin x$$

$\tilde{u} = n x^{n-1}$, $\tilde{v}' = \sin x$

$\tilde{v}' = \sin$, $\tilde{v} = -\cos$

$\tilde{u} \cdot \tilde{v} - \int \tilde{u}' \cdot \tilde{v}$

$$= x^n \cdot \sin x - n\left[x^{n-1} \cdot (-\cos x) - \int dx\, (n-1) x^{n-2} \cdot (-\cos x) \right]$$

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Bsp 3:

$v' = \sin x$, $u = e^x$

$$I = \int dx\; \sin x\; e^x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$v \cdot u - \int u' \cdot v$

$v' = \sin x$, $v = -\cos x$

$$= e^x \cdot (-\cos x) + \int dx\; e^x \cdot (+\cos x)$$

$\tilde{u} = e^x$, $\tilde{v}' = \cos x$

$\tilde{v}' = \cos x$, $\tilde{v} = \sin x$

$\tilde{u} \cdot \tilde{v} - \int \tilde{u}' \cdot \tilde{v}$

$$\int dx\; \sin x\; e^x = -\cos x\; e^x + \left[e^x \sin x - \int dx\; e^x \sin x \right]$$

$$\cancel{2} \int dx\; \sin x\; e^x = \frac{1}{2} e^x (\sin x - \cos x)$$

Check: $\frac{d}{dx} \frac{1}{2} e^x (\sin x - \cos x) = \frac{1}{2} e^x (\sin x - \cancel{\cos x}) + \frac{1}{2} e^x (+\cancel{\cos x} + \sin x)$ = ✓

$$= e^x \sin x$$ ✓.

4.6 Substitution

gegeben seien: $F'(x) = f(x)\,, \quad g(x)$ (1) (13)

$$\left[F(g(x))\right]' \overset{KR}{=} \underbrace{\underbrace{F'(g(x))}\cdot g'(x) = f\big(\underbrace{g(x)}\big)}_{(1)}\cdot g'(x) \qquad (2)$$

$\int dx\,(2)$:

$$\int_a^b dx\; f(g(x))\cdot g'(x) \overset{(2)}{=} \int_a^b dx\left[F(g(x))\right]' = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

$y = g(x)$

$y_b = g(b)$

$y_a = g(a)$

$$= F(y)\Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} dy\; f(y)$$

weil $F' \overset{(1)}{=} f$

$$\int_a^b dx\; f(g(x))\cdot \frac{d}{dx}g(x) = \int_{g(a)}^{g(b)} dy\; f(y) = \int_{y_a}^{y_b} dy\; f(y)$$

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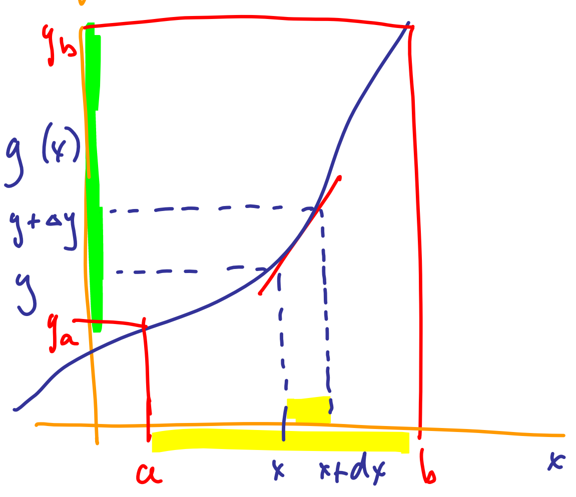
Merkregel: $g(x) = y$ Integrationsgrenzen: $g(b) = y_b$

$g(a) = y_a$

$$g'(x) = \frac{dg(x)}{dx} = \frac{dy}{dx} \qquad „dx\, g'(x) = dy“$$

$$\int d\cancel{x}\; \overbrace{f\big(\underbrace{g(x)}_{y}\big)}^{f(y)} \cdot \underbrace{g'(x)\, dx}_{dy} = \int_{y_a}^{y_b} dy\; f(y)$$



Beispiel 4: $I = \int dx \frac{1}{(2x+3)^2} = \frac{1}{2}\int dy \frac{1}{y^2} = -\frac{1}{2}\frac{1}{y}$ (15)

Subst: $y(x) = 2x+3$, $y(0) = 3$, $y(1) = 5$

$$\frac{dy}{dx} = 2 \Rightarrow \text{“} dy = 2\,dx \text{”} \Rightarrow \text{“} dx = \frac{1}{2} dy \text{”}$$

$$\tilde{I} = \int_0^1 dx \frac{1}{(2x+3)^2} = \frac{1}{2}\int_{y(0)}^{y(1)} dy \frac{1}{y^2} = \frac{1}{2}\int_3^5 dy \frac{1}{y^2}$$

$$= -\frac{1}{2}\frac{1}{y}\Big|_3^5 = -\frac{1}{2}\left[\frac{1}{5} - \frac{1}{3}\right]$$

4.6 Partialbruchzerlegung

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allgemeine Form

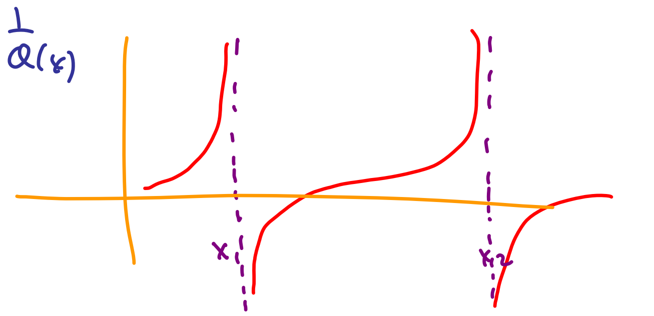
$$\text{Bsp:} \quad \int dx \frac{x+2}{x^2-5x+6} = \int dx \frac{R(x)}{Q(x)}$$

$R(x)$ ← Polynom 1. Grades

$Q(x)$ ← Polynom 2. Grades.

$$\text{Idee:} \quad \frac{R(x)}{(x-x_1)(x-x_2)} = \frac{R(x)}{Q(x)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2},$$

wobei x_1, x_2 die Nullstellen von $Q(x)$ sind.



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Bestimmung v. A_1, A_2:

Nullstellen v. $Q(x) = x^2 - 5x + 6 = (x-3)(x-2) \Rightarrow x_1 = 3$

$x_2 = 2$

$$\frac{R(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} = \frac{A_1}{x-3} + \frac{A_2}{x-2} = \frac{-4}{x-3} + \frac{5}{x-2}$$

$$\frac{R(x)}{(x-x_1)(x-x_2)} = \frac{A_1(x-x_2) + A_2(x-x_1)}{(x-x_1)(x-x_2)}$$

einsetzen.

$$R(x) = A_1(x-x_2) + A_2(x-x_1)$$

$$x+2 = A_1(x-2) + A_2(x-3)$$

auflösen nach

$\Rightarrow A_1 = 5$

$A_2 = -4$

$$x(\underbrace{1 - A_1 - A_2}_{=0}) = \underbrace{-2 - 2A_1 - 3A_2}_{=6}$$

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$$I = \int dx \left(\frac{5}{x-3} + \frac{-4}{x-2} \right) =$$

$$= 5 \ln|x-3| - 4 \ln|x-2|$$

Falls: $Q(x) = (x-x_1)^2 (x-x_2)(x-x_3)$

Ansatz: $$\frac{R(x)}{Q(x)} = \frac{A_1 x + B_1}{(x-x_1)^2} + \frac{B_2}{(x-x_1)} + \frac{A_2}{x-x_2} + \frac{A_3}{x-x_3}$$

R: Polynom n-ten Grades

Q: " n+1 - Grades.